Physics 129a
Hilbert Spaces
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1 Introduction

It is a fundamental postulate of quantum mechanics that any physically allowed state of a system may be described as a vector in a separable Hilbert space of possible states. Hilbert spaces figure prominently in the theory of differential equations. Loosely, a Hilbert space adds the notion of a scalar product to the ingredients of linearity and continuity. These notes collect and remind you of several definitions connected with the notion of a Hilbert space. We also illustrate the relation of a Hilbert space to more general spaces from which it is derived.

Two (of many, but I like these) references for further information are:

1. Guido Fano, *Mathematical Methods of Quantum Mechanics*, McGraw-Hill (1971). This is a text on the rigorous mathematical foundations of quantum mechanics. It is a very accessible read for the physics student.

2. John L. Kelley, *General Topology*, Van Nostrand Reinhold Company (1955). This is a textbook for mathematicians, but has lots of concise information for the impatient.

This note may largely be summarized by Fig. 1, showing the relationship between the different abstract spaces discussed.

2 Some Preliminaries

Definition: A **relation** is a set of ordered pairs.

Definition: A **function** is a relation such that no two distinct members have the same first coordinate. (To a mathematician, the following terms are synonymous: function, map, operator, transformation, correspondence).

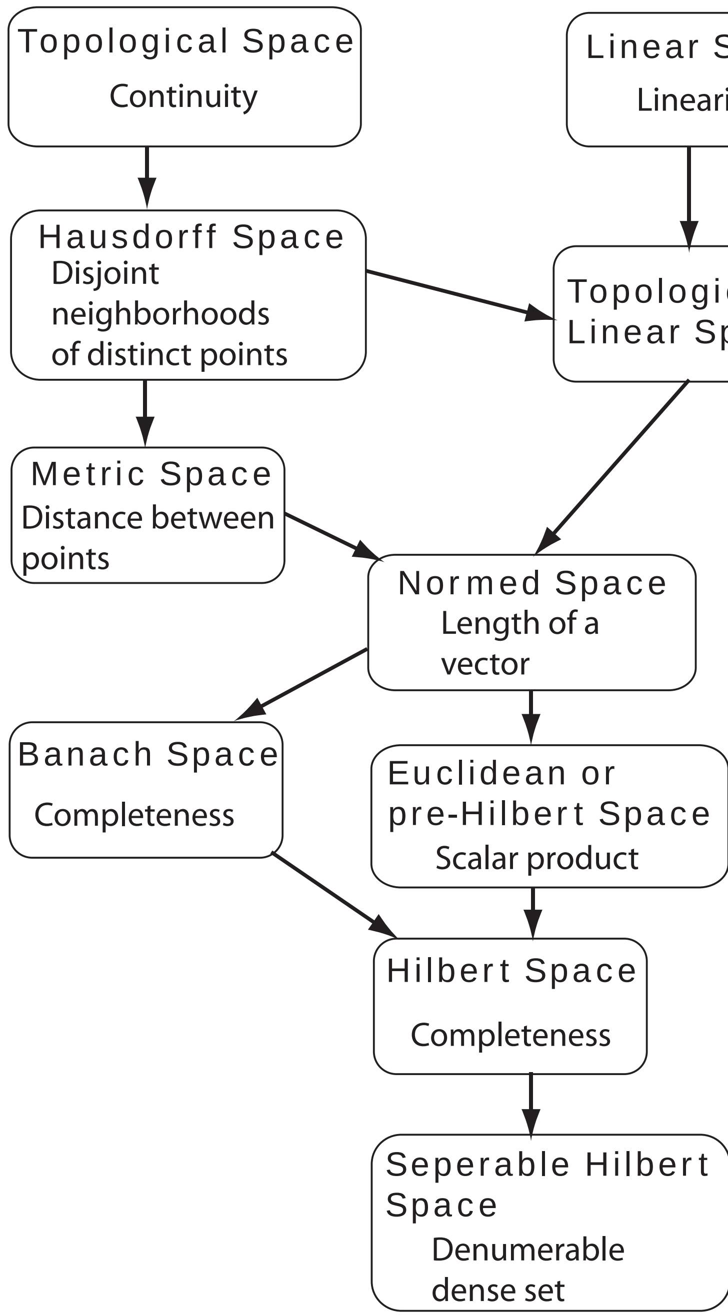

Figure 1: The relationships between several abstract spaces. Structure is added from top to bottom, that is the lower boxes inherit the properties of the upper ones to which they are connected, with additional structure noted in the box.

3 Continuity

The concept of a topological space embodies the notion of continuity in general.

Definition: A **topological space** T is a pair $(\mathcal{T}, \tau)$, where $\mathcal{T}$ is a non-empty set and τ is a family of subsets of $\mathcal{T}$ such that:

1. $\emptyset \in \tau$ and $\mathcal{T} \in \tau$,
2. the intersection of any two elements of τ is in τ, and
3. the arbitrary union of elements of τ is in τ.

The family τ is called a **topology** on $\mathcal{T}$, and its elements are called the **open sets** of $\mathcal{T}$.

There are some special names for particular cases. For example, given any set $\mathcal{T}$, the family $\tau = \{\emptyset, \mathcal{T}\}$ defines a topology on $\mathcal{T}$, called the **indiscrete topology**. Alternatively, the set of all subsets of $\mathcal{T}$ also defines a topology on $\mathcal{T}$, called the **discrete topology**. A familiar topological space is the space of real numbers ($\mathcal{R}$), with the **usual topology** generated by open intervals (also called the **ordinary topology** or the **Euclidean topology**). With the topology understood, we often refer to $\mathcal{T}$ as a topological space.

The notion of a topological space is very abstract. For example, the set $\mathcal{T}_4 = \{a, b, c, d\}$ with topology $\tau_4 = \{\emptyset, \mathcal{T}, \{a\}, \{b\}, \{a, b\}\}$ defines a topological space, as the reader should verify.

Definition: The complement (with respect to $\mathcal{T}$) of an open set is called a **closed set**.

Note that a set may be both open and closed, or may be neither open nor closed. For example, the two sets $\emptyset$ and $\mathcal{T}$ are always both open and closed in any topological space. In a discrete topology, every set is both open and closed. On the other hand, in $\mathcal{R}$ with the usual topology, the only sets that are both closed and open are $\emptyset$ and $\mathcal{T}$.

Definition: Given a point $t \in \mathcal{T}$, a set which contains an open set containing t is called a **neighborhood** of t.

Sometimes the neighborhood is defined to be an open set instead of a set containing an open set, but the essential feature is the same. The concept of a neighborhood provides the abstract notion of continuity.

Given any set A in a topological space, we may define the notion of its "closure", as the minimal closed set containing A. Intuitively, we add all the points on the "boundary" to get the closure:

Definition: Let A be a subset of $\mathcal{T}$, where topology τ has been imposed. The **closure** of A, denoted $\bar{A}$, is the intersection of all closed sets containing A.

For example, in the usual topology on $\mathcal{R}$ the closure of $(0, 1)$ is $[0, 1]$.

We futher define the idea of a set which contains elements that are arbitrarily close to (in the sense of topology τ) any given element of the topological space:

Definition: A subset A of a topological space $\mathcal{T}$ (with topology τ) is called **dense** if its closure is $\mathcal{T}$: $\bar{A} = \mathcal{T}$.

For example, in the usual topology on $\mathcal{R}$ the closure of the rationals, $\mathcal{Q}$ is $\mathcal{R}$, hence the rationals are dense in $\mathcal{R}$. Note that it is important to keep the topology in mind – if the discrete topology is used, then $\mathcal{Q}$ is not dense in $\mathcal{R}$.

The concept of continuity may now be introduced:

Definition: (**Continuous**) Given topological spaces T and S, and a mapping $f : \mathcal{T} \to \mathcal{S}$ from $\mathcal{T}$ into $\mathcal{S}$, f is called **continuous** at a point $t_0 \in \mathcal{T}$ if: For every neighborhood N_S of $f(t_0)$, there exists a neighborhood N_T of t_0 contained in $f^{-1}(N_S)$.

The set $f^{-1}(N_S)$ is called the inverse image of N_S. It consists of all points $x \in T$ such that $f(x) \in N_S$. This definition insures that, if a mapping is continuous, nearby points in the sense of the topology of $\mathcal{T}$, are mapped to nearby points in the sense of the topology on $\mathcal{S}$.

A particularly important class of topological spaces in physics is:

Definition: If for every pair of distinct points in a topological space there exist disjoint neighborhoods of the two points, the space is called a **Hausdorff space**.

In particular, the space of ordered n-tuples of real numbers, with the usual topology, is a Hausdorff space.

We may introduce the notion of "distance" between two points in a space:

Definition: (**Metric Space**) A non-empty set M is called a (positive-definite) **metric space** if to every pair of elements $x, y \in M$ there is given a real number $d(x, y)$ called the **distance** between x and y, such that:

1. $0 \leq d(x, y)$ (and $< \infty$)

2. $d(x, y) = 0 \quad \text{iff} \quad x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

The last condition should be recognized as the "triangle inequality".

It may be remarked that a metric space is a Hausdorff topological space, with topology generated by sets of the form $d(x, y) < r$, where r is a real number. It is left as an exercise to demonstrate this. We often use the suggestive notation $d(x, y) = |x - y|$.

The ability to measure distances provides the opportunity to define our familiar notions of convergence.

Definition: A sequence of elements in a metric space, $x_1, x_2, \cdots$ is said to **converge** to an element x if, given $\epsilon > 0$, there exists a number N such that: $d(x, x_n) < \epsilon$ whenever $n > N$. In this case, we write $x = \lim_{n\to\infty} x_n$.

Definition: A sequence of elements in a metric space is called a **Cauchy sequence** if, given $\epsilon > 0$, there exists N such that:

$$| x_n - x_m | < \epsilon \qquad \text{for all } n, m > N.$$

("Cauchy Convergence Criterion")

For example, the sequence $x_n = 1/n, n = 1, 2, \ldots$ is a Cauchy sequence, converging to 0 (in the metric space of the real numbers with $d(x, y) = |x - y|$, that is with the usual topology). We will henceforth assume the usual topology and distance function unless otherwise specified.

Theorem: Every convergent sequence of elements in a metric space is a Cauchy sequence.

Proof: Suppose $x_1, x_2, \cdots$ is a convergent sequence of elements such that $\lim_{n\to\infty} x_n = x$. Then, given any $\epsilon > 0$, we may find a number N such that:

$$d(x, x_n) < \frac{1}{2}\epsilon \qquad \text{for all } n > N$$

Consider

$$\begin{aligned}
|x_n - x_m| &= |x_n - x + x - x_m| && (1)\\
&\leq |x_n - x| + |x - x_m| \qquad \text{(triangle inequality)} && (2)\\
&< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \qquad \text{for all } n, m > N && (3)\\
&< \epsilon && (4)
\end{aligned}$$

QED

Definition: A metric space $(M,\ d)$ is said to be **complete** if every Cauchy sequence of points in V converges to a point in V.

For example, the real numbers form a complete metric space.

Theorem: (and Definition) If $(M,\ d)$ is an incomplete metric space, there exists a complete metric space $(M^*,\ d^*)$ called the **completion** of M which corresponds to an isometric (*i.e.*, distance preserving) mapping of M into M^* such that the closure (*i.e.*, the intersection of all closed sets containing M) of the image of M coincides with M^*. (For an instructive, but mildly lengthy proof, see: Fano, Mathematical Methods of Quantum Mechanics.)

The rationals admit a completion by adding the irrationals, i.e., $M^* = \mathcal{R}$ and $d^* = d$.

We may define notions of compactness and boundedness for subsets of a metric space:

Definition: Let B be a subset of a metric space (M, d). B is called **bounded** if, for some $x \in M$ there exists a real number r such that $d(b, x) < r$ for all $b \in B$.

Definition: Let C be a subset of a metric space (M, d). C is called **relatively compact** if every infinite sequence of distinct elements of C has a convergent subsequence. If the subsequence converges to an element of C, then C is said to be **compact**.

Note that if C is relatively compact, then $\bar{C}$ is compact. The Bolzano-Weierstrass theorem tells us that every closed bounded set in the metric space R^n, with the usual topology, is compact. For example, $[0, 1]$ is bounded and compact, while $(0, 1)$ is bounded and relatively compact, but not compact.

4 Linearity

We see that the notion of a topological space embodies spaces with an intuitive idea of a distance. However, another ingredient, "linearity" is required to develop the richness of the spaces we commonly entertain in physical theory.

Definition: A (nonempty) set V is called a (**complex**) **linear** (or **vector**) **space**, and its elements are called **vectors** if:

(a) An operation of "addition" is defined for every pair of elements $x \in V$ and $y \in V$, such that the "sum" $x + y \in V$.

(b) There exists a "zero vector," $0 \in V$ such that

$$x + 0 = x \qquad \text{for all x } \epsilon \text{ V}$$

(c) An operation of "multiplication" by a complex number ("scalar") is defined so that, if c is any complex number, and x any vector in V, then the "product" $cx \in V$.

(d) The following properties (of ordinary vector algebra) are satisfied: (x, y, and z are any elements of V, and c, c_1, and c_2 are any complex numbers)

$x + y = y + x$	1) commutativity
$(x + y) + z = x + (y + z)$	2) associativity
$c_1(c_2x) = (c_1c_2)x$	
$x + (-x) = 0$	3) inverse
$1x = x$	4) multiplication by scalar 1
$(c_1 + c_2)x = c_1x + c_2x$	5) distributivity
$c(x + y) = cx + cy.$	

If we restrict to real scalars, we have a "real vector space." Alternatively, we could generalize the space of scalars (to arbitrary fields), but we need not pursue that here. Note that the 0 element is unique by virtue of the fact that the vectors under the operation of addition form an abelian group (anticipating our study of group theory next quarter).

Definition: m vectors $x^{(1)}, x^{(2)}, \cdots, x^{(m)}$ are **linearly dependent** if there exist m constants, not all zero, such that

$$\sum_{i=1}^{m} c_i x^{(i)} = 0$$

Otherwise, the vectors are **linearly independent.**

Definition: A linear space V is n-**dimensional** if it contains a set of n linearly independent vectors, but no set of more than n linearly independent vectors.

Definition: A set of linearly independent vectors, $e_1, e_2, e_3, \cdots$ in a vector space V forms a **basis** for V if, for any vector $x \;\epsilon\; V$, we can find scalars $c_1(x), c_2(x), \cdots$ such that:

$$x = c_1(x)e_1 + c_2(x)e_2 + \cdots$$

5 Topological Linear Spaces

We are ready now to combine the notions of continuity and linearity into a common structure called a topological linear space.

Definition: (**Topological Linear Space**) Let U be a linear space upon which a topology has been defined. Then U is a **topological linear space** if

1. U is a Hausdorff space.
2. The function that sums any two vectors in U is a continuous function of both vectors. That is, $x + y$ is continuous in x and y.
3. Multiplication of a vector x by scalar c is a continuous function of both x and c.

The reader is invited to verify that some familiar spaces, with familiar topologies, satisfy these conditions. Note that the continuity under multiplication is both in the sense that cx is a continuous mapping from the complex numbers, $\mathcal{C}$ to vector space L, under multiplication of c by vector x, and a continuous mapping from vector space L to vector space L under multiplication of vector x by complex number c.

We introduce now the idea of the length or "norm" of a vector:

Definition: (**Normed Space**) Given a linear space L, if a mapping $|\cdots|$ from L into the non-negative real numbers is defined such that:

1. $|x| = 0$ iff $x = 0$,
2. $|cx| = \text{abs}(c)|x|$, $\forall x \in L$ and for all scalars c,
3. the triangle inequality is satisfied: $|x + y| \leq |x| + |y|$, $\forall x, y \in L$,

then L is called a normed space with norm $||$.

We notice that a normed space is also a metric space, simply define the distance between two elements to be $d(x, y) = |x - y|$ (hence, our use of this notation already). The reader should check that the desired properties hold. It may be remarked, however, that a metric space is not necessarily a normed space, as a metric space is not required to be a linear space.

We may prove that a normed space is a topological linear space. We already know that it is a linear space. It is also a Hausdorff space since it is a metric space. It remains to demonstrate the continuity properties of a topological linear space. For example, let us demonstrate the continuity of the cx mapping. Consider a neighborhood of a point x_0:

$$N(x_0; \epsilon) \equiv \{x : |x - x_0| < \epsilon\}. \tag{5}$$

If $x \in N(x_0; \epsilon)$ then cx is in the neighborhood:

$$N(cx_0; |c|\epsilon) = \{x : |x - cx_0| < |c|\epsilon\}. \tag{6}$$

This is true since

$$|cx - cx_0| = |c(x - x_0)| = c|x - x_0| < |c|\epsilon. \tag{7}$$

The continuity of $x + y$ may be demonstrated similarly.

We may here inject the property of completeness:

Definition: A normed space that is complete is called a **Banach Space**.

Alternatively, we may introduce the idea of a scalar product of two vectors[1]:

Definition: A linear space V is called **pre-Hilbert**, or **Euclidean**, if a function is defined which assigns to every pair of vectors $x, y \,\epsilon\, V$ a complex number $\langle x|y\rangle$, called the **scalar product** (or **inner product**) of x and y, which satisfies the following properties:

1. $\langle x|x\rangle \geq 0; \quad \langle x|x\rangle = 0 \quad \text{iff} \quad x = 0$
2. $\langle x|y\rangle = \langle y|x\rangle^*$
3. $\langle x|cy\rangle = c\langle x|y\rangle \qquad$ (c is any complex number)
4. $\langle x|y_1 + y_2\rangle = \langle x|y_1\rangle + \langle x|y_2\rangle$

Note that a metric space need not be a linear space. However, if we have a pre-Hilbert space, we may define a suitable distance according to:

$$d(x, y) = \sqrt{\langle (x - y)|(x - y)\rangle}$$

We will typically deal only with metric spaces which are also pre-Hilbert spaces. As before, we will also use the notation

$$|x - y| = d(x, y)$$

[1]Here we define a "positive definite" scalar product, in which the scalar product of a vector with itself is non-negative, and zero only for the zero vector. In some cases, for example relativity, this requirement is relaxed.

for the distance function. We will furthermore define the "length," or "norm," of a vector by its distance from the zero vector: $|x| = |x - 0| = d(x, 0) = \sqrt{\langle x|x\rangle}$. Also, if $\langle x|y\rangle = 0$, we say that x is "orthogonal" to y. We see that a pre-Hilbert space is also a normed space, hence a toplogical linear space.

Earlier we stated a theorem that said there exists a completion for any incomplete metric space. Thus, every pre-Hilbert space admits a completion, which we call a **Hilbert space**.

We said at the beginning that it is a fundamental postulate of quantum mechanics that every physical state corresponds to a vector in a separable Hilbert space. To complete the understanding of the meaning of this, it only remains to define what we mean by "separable".

Definition: A separable Hilbert space is one which contains a denumerable dense set.

This gives us a structure such that we may consider sets of basis vectors which are countable. That is, arbitrary quantum mechanical states can be constructed from a linear combination of basis states in a sum with integer index.

If we wish to continue this discussion, for example to construct a suitable Hilbert space of functions for quantum mechanical problems, we will properly need to consider measure theory, *etc.* However, we instead turn briefy to other matters here.

6 Operators

Definition: Suppose that we have two vector spaces, V and V', and that there exists a correspondence which assigns to every vector $x \,\epsilon\, D_A \subset V$, a vector $x' \epsilon\, V'$. We say that this correspondence defines an **operator** A from V into V' with **domain** D_A, and write $x' = Ax$.

The subset R_A of V' defined by

$$R_A = \{x' \mid x' = Ax \text{ for some } x \epsilon D_A\}$$

is called the **range** of A.

If $V' = V$, then we say that A is defined **in** V.

If $D_A = V$, then we say that A is defined **on** V.

If V' is the vector space formed by the complex numbers (with ordinary operations of addition and multiplication by a complex number), we often use the term **functional** instead of operator, and write $x' = f(x)$ rather than $x' = Ax$.

Definition: Two operators A and B from V into V' are said to be **equal** if $D_A = D_B$ and

$$Ax = Bx \qquad \text{for all } x \in D_A$$

If, on the other hand, D_B is a proper subset of D_A, and $Ax = Bx$ for all $x \in D_B$, we call A an **extension** of B, and B a **restriction** of A. We may denote this by writing $B \subset A$.

Definition: An operator L is said to be **linear** if its domain D_L is a subspace of V (*i.e.*, D_L is a vector space) and

$$L(x+y) = Lx + Ly \qquad \text{for all } x,\ y \in D_L$$

$$L(cx) = c\,Lx \qquad \text{for all } x \in D_L \text{ and for all scalars c}$$

Henceforth, we will typically mean "linear operator" whenever we say "operator".

There are certain types of operators with especially "nice" properties:

Definition: An operator A is called **bounded** if there exists a real number c such that

$$|Ax| < c|x|, \quad \forall x. \tag{8}$$

Note that the norms are with respect to the two normed spaces involved, even though the same notation is used. For example, linear operators on a finite-dimensional space (matrices) are bounded.

Definition: An operator on a Banach space is called **compact** or **completely continuous** if it maps bounded sets into relatively compact sets.

A compact operator is also bounded.

7 Exercises

1. Prove: Given a topological space $(\mathcal{T}, \tau)$ and a set $A \subseteq \mathcal{T}$, then A is open if and only if it contains a neighborhood of each of its points. Note: This is easy, but make sure you really do prove it!

2. Which of the following define topological spaces:

   (a) The set $\mathcal{T} = \{1,2,3,4,5,6,7,8,9\}$, with (proposed) topology $\tau = \{\emptyset, \{1,2,3,4,5\}, \{6,7,8,9\}, \mathcal{T}\}$.

   (b) The set $\mathcal{T} = \{1,2,3,4,5,6,7,8,9\}$, with (proposed) topology $\tau = \{\emptyset, \{1,2,3,4,5\}, \{5,6\}, \{6,7,8,9\}, \mathcal{T}\}$.

   (c) The set of integers, $\mathcal{I} = \{0, \pm 1, \pm 2, \ldots\}$, with (proposed) topology $\tau = \{\emptyset, \{\text{even integers}\}, \{\text{odd integers}\}\}$.

   (d) The set of real numbers $\mathcal{R}$, with (proposed) topology τ given by sets which are finite intersections and arbitrary unions of closed intervals.

   (e) The set of $n \times n$ Hermitian matrices, with (proposed) topology generated by imposing the usual topology on the real and imaginary parts of each of the elements of the matrices.

   (f) The set of rotations in three dimensions, with (proposed) topology generated by the usual topology on the Euler angles parameterizing the rotation.

3. Show that a metric space is also a Hausdorff space, with topology generated by sets of the form $d(x,y) < r$.

4. Complete the proof that a normed space is a topological linear space.

5. Consider the real vector space of real continuous functions with continuous first derivatives defined on the closed interval $[0,1]$. Which of the following defines a scalar product?

   (a)
   $$\langle f|g\rangle = \int_0^1 f'(x)g'(x)dx + f(0)g(0) \tag{9}$$

   (b)
   $$\langle f|g\rangle = \int_0^1 f'(x)g'(x)dx \tag{10}$$

6. Show that any $n \times n$ matrix M may be represented as an n-term dyad:
   $$M = \sum_{i=1}^{n} |a_i\rangle\langle b_i|, \tag{11}$$
   where $|a_i\rangle$ and $|b_i\rangle$ are n-dimensional complex vectors. Thus, every linear operator in E_n (n-dimensional Euclidean space) may be expressed as an n-term dyad.

7. Consider the following equation in E_∞ (infinite-dimensional Euclidean space; let the scalar product be $\langle x|y\rangle = \sum_{n=1}^{\infty} x_n^* y_n$.):

$$Cx = a, \tag{12}$$

where the operator is defined in some basis by:

$$C(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots). \tag{13}$$

(a) Is C a bounded operator? That is, does there exist a non-negative real number B such that

$$|Cx| \leq B|x|, \quad \forall x \in E_\infty? \tag{14}$$

(b) Is C a linear operator?

(c) Is C a Hermitian operator?

(d) Does $Cx = 0$ have a non-trivial solution?

(e) Does $Cx = a$ always have a solution?

Now answer the same questions for the operator defined by:

$$G(x_1, x_2, x_3, \ldots, x_n, \ldots) = (x_1, x_2/2, x_3/3, \ldots, x_n/n, \ldots). \tag{15}$$

Note that we require a vector to be normalizable if it is to belong to E_∞ – the scalar product of a vector with itself must exist.