

Linear Differential Equations
Physics 129a
Solutions to Problems
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1 Exercises

1. Consider the general linear second order homogeneous differential equation in one dimension:

$$a(x)\frac{d^2}{dx^2}u(x) + b(x)\frac{d}{dx}u(x) + c(x)u(x) = 0. \quad (1)$$

Determine the conditions under which this may be written in the form of a differential equation involving a self-adjoint (with appropriate boundary conditions) **Sturm-Liouville operator**:

$$Lu = 0, \quad (2)$$

where

$$L = \frac{d}{dx}p(x)\frac{d}{dx} - q(x). \quad (3)$$

2. Show that the operator

$$L = \frac{d^2}{dx^2} + 1, \quad x \in [0, \pi], \quad (4)$$

with homogeneous boundary conditions $u(0) = u(\pi) = 0$, is self-adjoint.

3. Let us consider somewhat further the “momentum operator”, $p = \frac{1}{i}\frac{d}{dx}$, discussed briefly in the differential equation note. We let this operator be an operator on the Hilbert space of square-integrable (normalizable) functions, with $x \in [a, b]$.
 - (a) Find the most general boundary condition such that p is Hermitian.
 - (b) What is the domain, D_p , of p such that p is self-adjoint?
 - (c) What is the situation when $[a, b] \rightarrow [-\infty, \infty]$? Is p bounded or unbounded?

4. Prove that the different systems of orthogonal polynomials are distinguished by the weight function and the interval. That is, the system of polynomials in $[a, b]$ is uniquely determined by $w(x)$ up to a constant for each polynomial.
5. We said that the recurrence relation for the orthogonal polynomials may be expressed in the form:

$$f_{n+1}(x) = (a_n + b_n x) f_n(x) - c_n f_{n-1}(x), \quad (5)$$

see Eqn. 154. Try to verify.

6. We discussed some theorems for the qualitative behavior of classical orthogonal polynomials, and illustrated this with the one-electron atom radial wave functions. Now consider the simple harmonic oscillator (in one dimension) wave functions. The potential is

$$V(x) = \frac{1}{2} k x^2. \quad (6)$$

Thus, the Schrödinger equation is

$$-\frac{1}{2m} \frac{d^2}{dx^2} \psi(x) + \frac{1}{2} k x^2 \psi(x) = E \psi(x). \quad (7)$$

Make a sketch showing the qualitative features you expect for the wave functions corresponding to the five lowest energy levels.

Try to do this with some care: There is really a lot that you can say in qualitative terms without ever solving the Schrödinger equation. Include a curve of the potential on your graph. Try to illustrate what happens at the classical turning points (that is, the points where $E = V(x)$).

7. Find the Green's function for the operator

$$L = \frac{d^2}{dx^2} + k^2, \quad (8)$$

where k is a constant, and with boundary conditions $u(0) = u(1) = 0$. For what values of k does your result break down?

8. An integral that is encountered in calculating radiative corrections in e^+e^- collisions is of the form:

$$I(t; a, b) = \int_a^b \frac{x^{t-1}}{1-x} dx, \quad (9)$$

where $0 \leq a < b \leq 1$, and $t \geq 0$.

Show that this integral may be expressed in terms of the hypergeometric function ${}_2F_1$. Make sure to check the $t = 0$ case.

9. We will do Mathews and Walker problem 9-3 by two methods and compare. We'll do one method this week, another method next week. Thus, we consider the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \quad (10)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Such a situation may arise, for example, if we are interested in the electric field inside a conducting sphere. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (11)$$

such that $G(r = a, \mathbf{y}) = 0$.

Find $G(\mathbf{x}, \mathbf{y})$ by obtaining solutions to the homogeneous equation

$$(\nabla^2 + k^2)G = 0, \quad (12)$$

on either side of $r = |\mathbf{y}|$; satisfying the boundary conditions at $r = a$, and the appropriate matching conditions at $r = |\mathbf{y}|$.

Solution: The symmetry of the problem is such that it is convenient to work in spherical polar coordinates.

Let $r = |\mathbf{x}|$ and $r' = |\mathbf{y}|$. Let $r_<$ be the smaller of (r, r') and let $r_>$ be the larger of (r, r') . Let (θ, ϕ) be the polar and azimuthal angles of \mathbf{x} , and likewise let (θ', ϕ') be the polar and azimuthal angles of \mathbf{y} . Let ψ be the angle between \mathbf{x} and \mathbf{y} , that is,

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad (13)$$

The final answer is:

$$G(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \frac{k(2\ell + 1)P_{\ell}(\cos \psi)}{4\pi} \left\{ \frac{j_{\ell}(kr_<)}{j_{\ell}(ka)} [j_{\ell}(ka)n_{\ell}(kr_>) - n_{\ell}(ka)j_{\ell}(kr_>)] \right\}. \quad (14)$$

10. We return to Mathews and Walker problem 9-3, that is the preceding problem. This is the problem of the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \quad (15)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Such a situation may arise, for example, if we are interested in the electric field inside a conducting sphere. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (16)$$

such that $G(r = a, \mathbf{y}) = 0$.

Last week, you found $G(\mathbf{x}, \mathbf{y})$ by obtaining solutions to the homogeneous equation

$$(\nabla^2 + k^2)G = 0, \quad (17)$$

on either side of $r = |\mathbf{y}|$; satisfying the boundary conditions at $r = a$, and the appropriate matching conditions at $r = |\mathbf{y}|$.

Now we take a different approach: Find G by directly solving $(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$. You should ignore the boundary conditions at first and obtain a solution by integrating the equation over a small volume containing \mathbf{y} . Then satisfy the boundary conditions by adding a suitable function $g(\mathbf{x}, \mathbf{y})$ that satisfies $(\nabla_x^2 + k^2)g(\mathbf{x}, \mathbf{y}) = 0$ everywhere.

11. Let's continue our discussion of Mathews and Walker problem 9-3, that is the discussion in the preceding two problems. This is the problem of the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \quad (18)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Such a situation may arise, for example, if we are interested in the electric field inside a conducting sphere. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (19)$$

such that $G(r = a, \mathbf{y}) = 0$.

Last week, you found $G(\mathbf{x}, \mathbf{y})$ by directly solving $(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, ignoring the boundary conditions at first. This is called the "fundamental solution" because it contains the desired singularity structure, and hence has to do with the "source". Now find the fundamental solution by another technique: Put the origin at \mathbf{y} and solve the equation

$$(\nabla_x^2 + k^2)f(\mathbf{x}) = \delta(\mathbf{x}), \quad (20)$$

by using Fourier transforms. Do you get the same answer as last week?

12. Referring still to Mathews and Walker problem 9-3, discuss the relative merits of the solutions found in problems 9 and 10. In particular, analyze, by making a suitable expansion, a case where the problem 10 solution is likely to be preferred, stating the necessary assumptions clearly.
13. We noted that the Green's function method is applicable beyond the Sturm-Liouville problem. For example, consider the differential operator:

$$L = \frac{d^4}{dx^4} + \frac{d^2}{dx^2}. \quad (21)$$

As usual, we wish to find the solution to $Lu = -\phi$. Let us consider the case of boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$.

- (a) Find the Green's function for this operator.
 (b) Find the solution for $x \in [0, \infty]$ and $\phi(x) = e^{-x}$.

You are encouraged to notice, at least in hindsight, that you could probably have solved this problem by elementary means.

14. Using the Green's function method, we derived in class the time development transformation for the free-particle Schrödinger equation in one dimension:

$$U(x, y; t) = \frac{1}{\sqrt{2}} \left(1 - i \frac{t}{|t|} \right) \sqrt{\frac{m}{2\pi|t|}} \exp \left[\frac{im(x-y)^2}{2t} \right]. \quad (22)$$

This should have the property that if you do a transformation by time t , followed by a transformation by time $-t$, you should get back to where you started. Check whether this is indeed the case or not.

15. Using the Christoffel-Darboux formula, find the projection operator onto the subspace spanned by the first three Chebyshev polynomials.
16. We discussed the radial solutions to the “one-electron” Schrödinger equation. Investigate orthogonality of the result – are our wave functions orthogonal or not?
17. In class we considered the problem with the Hamiltonian

$$H = -\frac{1}{2m} \frac{d^2}{dx^2}. \quad (23)$$

Let us modify the problem somewhat and consider the configuration space $x \in [a, b]$ (“infinite square well”).

- (a) Construct the Green's function, $G(x, y; z)$ for this problem.
- (b) From your answer to part (a), determine the spectrum of H .
- (c) Notice that, using

$$G(x, y; z) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k^*(y)}{\omega_k - z}, \quad (24)$$

the normalized eigenstate, $\phi_k(x)$, can be obtained by evaluating the residue of G at the pole $z = \omega_k$. Do this calculation, and check that your result is properly normalized.

- (d) Consider the limit $a \rightarrow -\infty$, $b \rightarrow \infty$. Show, in this limit that $G(x, y; z)$ tends to the Green's function we obtained in class for this Hamiltonian on $x \in (-\infty, \infty)$:

$$G(x, y; z) = i\sqrt{\frac{m}{2z}}e^{i\rho|x-y|}. \quad (25)$$

18. Let us investigate the Green's function for a slightly more complicated situation. Consider the potential:

$$V(x) = \begin{cases} V & |x| \leq \Delta \\ 0 & |x| > \Delta \end{cases} \quad (26)$$

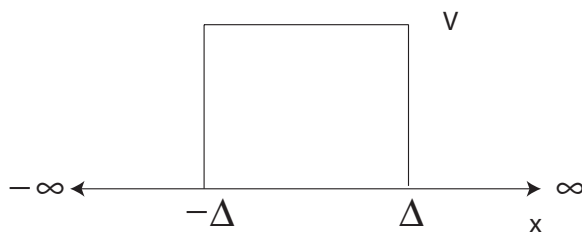


Figure 1: The “finite square potential”.

- (a) Determine the Green's function for a particle of mass m in this potential.

Remarks: You will need to construct your “left” and “right” solutions by considering the three different regions of the potential, matching the functions and their first derivatives at the boundaries. Note that the “right” solution may be very simply obtained

from the “left” solution by the symmetry of the problem. In your solution, let

$$\rho = \sqrt{2m(z - V)} \quad (27)$$

$$\rho_0 = \sqrt{2mz}. \quad (28)$$

Make sure that you describe any cuts in the complex plane, and your selected branch. You may find it convenient to express your answer to some extent in terms of the force-free Green’s function:

$$G_0(x, y; z) = \frac{im}{\rho} e^{i\rho_0|x-y|}. \quad (29)$$

- (b) Assume $V > 0$. Show that your Green’s function $G(x, y; z)$ is analytic in your cut plane, with a branch point at $z = 0$.
- (c) Assume $V < 0$. Show that $G(x, y; z)$ is analytic in your cut plane, except for a finite number of simple poles at the bound states of the Hamiltonian.
19. In class, we obtained the free particle propagator for the Schrödinger equation in quantum mechanics:

$$U(x, t; x_0, t_0) = \frac{1}{\sqrt{2}} \left(1 - i \frac{t - t_0}{|t - t_0|} \right) \sqrt{\frac{m}{2\pi|t - t_0|}} \exp \left[\frac{im(x - x_0)^2}{2(t - t_0)} \right]. \quad (30)$$

Let’s actually use this to evolve a wave function. Thus, let the wave function at time $t = t_0 = 0$ be:

$$\psi(x_0, t_0 = 0) = \left(\frac{1}{\pi a^2} \right)^{1/4} \exp \left(-\frac{x_0^2}{2a^2} + ip_0 x_0 \right), \quad (31)$$

where a and p_0 are real constants. Since the absolute square of the wave function gives the probability, this wave function corresponds to a Gaussian probability distribution (i.e., the probability density function to find the particle at x_0) at $t = t_0$:

$$|\psi(x_0, t_0)|^2 = \left(\frac{1}{\pi a^2} \right)^{1/2} e^{-\frac{x_0^2}{a^2}}. \quad (32)$$

The standard deviation of this distribution is $\sigma = a/\sqrt{2}$. Find the probability density function, $|\psi(x, t)|^2$, to find the particle at x at some later (or earlier) time t . You are encouraged to think about the “physical” interpretation of your result.

20. In class we considered the application of the Rayleigh-Ritz method to finding approximate eigenvalues satisfying

$$y'' = -\lambda y, \quad (33)$$

with boundary conditions $y(-1) = y(1) = 0$. Repeat the method, now with two functions:

$$\alpha_1(x) = 1 - x^2, \quad (34)$$

$$\alpha_2(x) = x^2(1 - x^2). \quad (35)$$

You should get estimates for two eigenvalues. Compare with the exact eigenvalues, including a discussion of which eigenvalues you have managed to approximate and why. If the eigenvalues you obtain are not the two lowest, suggest another function you might have used to get the lowest two.

21. The Bessel differential equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(k^2 - \frac{m^2}{x^2} \right) y = 0. \quad (36)$$

A solution is $y(x) = J_m(kx)$, the m th order Bessel function. Assume a boundary condition $y(1) = 0$. That is, k is a root of $J_m(x)$. Use the Rayleigh-Ritz method to estimate the first non-zero root of $J_3(x)$. I suggest you try to do this with one test function, rather than a sum of multiple functions. But you must choose the function with some care. In particular, note that J_3 has a third-order root at $x = 0$. You should compare your result with the actual value of 6.379. If you get within, say, 15% of this, declare victory.