

**Linear Differential Equations**  
**Physics 129a**  
**Solutions to Problems**  
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## 1 Exercises

1. Consider the general linear second order homogeneous differential equation in one dimension:

$$a(x)\frac{d^2}{dx^2}u(x) + b(x)\frac{d}{dx}u(x) + c(x)u(x) = 0. \quad (1)$$

Determine the conditions under which this may be written in the form of a differential equation involving a self-adjoint (with appropriate boundary conditions) **Sturm-Liouville operator**:

$$Lu = 0, \quad (2)$$

where

$$L = \frac{d}{dx}p(x)\frac{d}{dx} - q(x). \quad (3)$$

2. Show that the operator

$$L = \frac{d^2}{dx^2} + 1, \quad x \in [0, \pi], \quad (4)$$

with homogeneous boundary conditions  $u(0) = u(\pi) = 0$ , is self-adjoint.

**Solution:** Consider

$$\begin{aligned} \langle Lv|u \rangle - \langle v|Lu \rangle &= \int_0^\pi \left\{ \left[ \frac{d^2}{dx^2} + 1 \right] v(x) \right\}^* u(x) dx - \int_0^\pi v^*(x) \left[ \frac{d^2}{dx^2} + 1 \right] u(x) dx \\ &= \int_0^\pi \frac{d^2 v^*}{dx^2}(x) u(x) dx - \int_0^\pi v^*(x) \frac{d^2 u}{dx^2}(x) dx \\ &= \left. \frac{dv^*}{dx}(x) u(x) \right|_0^\pi - v^*(x) \left. \frac{du}{dx}(x) \right|_0^\pi \end{aligned} \quad (5)$$

The first term vanishes if  $u$  satisfies the boundary conditions, and the second term vanishes if  $v$  satisfies the boundary conditions. Thus,  $L$  is self-adjoint.

3. Let us consider somewhat further the “momentum operator”,  $p = \frac{1}{i} \frac{d}{dx}$ , discussed briefly in the differential equation note. We let this operator be an operator on the Hilbert space of square-integrable (normalizable) functions, with  $x \in [a, b]$ .
  - (a) Find the most general boundary condition such that  $p$  is Hermitian.
  - (b) What is the domain,  $D_P$ , of  $p$  such that  $p$  is self-adjoint?
  - (c) What is the situation when  $[a, b] \rightarrow [-\infty, \infty]$ ? Is  $p$  bounded or unbounded?
4. Prove that the different systems of orthogonal polynomials are distinguished by the weight function and the interval. That is, the system of polynomials in  $[a, b]$  is uniquely determined by  $w(x)$  up to a constant for each polynomial.
5. We said that the recurrence relation for the orthogonal polynomials may be expressed in the form:

$$f_{n+1}(x) = (a_n + b_n x) f_n(x) - c_n f_{n-1}(x), \quad (6)$$

see Eqn. 16. Try to verify.