

Linear Differential Equations
Physics 129a
Solutions to Problems
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1 Exercises

1. Consider the general linear second order homogeneous differential equation in one dimension:

$$a(x) \frac{d^2}{dx^2} u(x) + b(x) \frac{d}{dx} u(x) + c(x) u(x) = 0. \quad (1)$$

Determine the conditions under which this may be written in the form of a differential equation involving a self-adjoint (with appropriate boundary conditions) **Sturm-Liouville operator**:

$$Lu = 0, \quad (2)$$

where

$$L = \frac{d}{dx} p(x) \frac{d}{dx} - q(x). \quad (3)$$

Note that part of the problem is to investigate self-adjointness.

Solution: Probably the simplest way to approach this is to consider an “integrating factor” (e.g., Mathews and Walker, chapter 1; Riley, Hobson, and Bence, chapters 14 and 15): Assuming $a(x) \neq 0$, divide the equation by $a(x)$:

$$\frac{d^2}{dx^2} u(x) + \frac{b(x)}{a(x)} \frac{d}{dx} u(x) + \frac{c(x)}{a(x)} u(x) = 0. \quad (4)$$

Then multiply the equation by integrating factor $\exp[\int b(x)/a(x) dx]$:

$$e^{\int b(x)/a(x) dx} \frac{d^2}{dx^2} u(x) + e^{\int b(x)/a(x) dx} \frac{b(x)}{a(x)} \frac{d}{dx} u(x) + e^{\int b(x)/a(x) dx} \frac{c(x)}{a(x)} u(x) = 0. \quad (5)$$

This gives Sturm-Liouville form with

$$p(x) = e^{\int b(x)/a(x) dx} \quad (6)$$

$$q(x) = -e^{\int b(x)/a(x) dx} \frac{c(x)}{a(x)}. \quad (7)$$

To investigate self-adjointness of the Sturm-Liouville operator, we will assume that p and q are real. Consider

$$\langle Lv|u\rangle - \langle v|Lu\rangle = \int_a^b \left\{ \left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right] v(x) \right\}^* u(x) dx \quad (8)$$

$$- \int_a^b v^*(x) \left\{ \left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right] u(x) \right\} dx \quad (9)$$

$$= \int_a^b \left\{ \left[\frac{d}{dx} p(x) \frac{d}{dx} \right] v^*(x) \right\} u(x) dx - \int_a^b v^*(x) \left\{ \left[\frac{d}{dx} p(x) \frac{d}{dx} \right] u(x) \right\} dx$$

$$= p(x) \frac{dv^*(x)}{dx} u(x) \Big|_a^b - \int_a^b p(x) \frac{dv^*(x)}{dx} \frac{du(x)}{dx} dx \quad (10)$$

$$- v^*(x) p(x) \frac{du(x)}{dx} \Big|_a^b + \int_a^b \frac{dv^*(x)}{dx} p(x) \frac{du(x)}{dx} dx \quad (11)$$

$$= p(x) \frac{dv^*(x)}{dx} u(x) \Big|_a^b - v^*(x) p(x) \frac{du(x)}{dx} \Big|_a^b \quad (12)$$

The boundary conditions must be such that the last line is zero for L to be self-adjoint.

2. Show that the operator

$$L = \frac{d^2}{dx^2} + 1, \quad x \in [0, \pi], \quad (13)$$

with homogeneous boundary conditions $u(0) = u(\pi) = 0$, is self-adjoint.

Solution: Consider

$$\begin{aligned} \langle Lv|u\rangle - \langle v|Lu\rangle &= \int_0^\pi \left\{ \left[\frac{d^2}{dx^2} + 1 \right] v(x) \right\}^* u(x) dx - \int_0^\pi v^*(x) \left[\frac{d^2}{dx^2} + 1 \right] u(x) dx \\ &= \int_0^\pi \frac{d^2 v^*(x)}{dx^2} u(x) dx - \int_0^\pi v^*(x) \frac{d^2 u(x)}{dx^2} dx \\ &= \frac{dv^*(x)}{dx} u(x) \Big|_0^\pi - v^*(x) \frac{du(x)}{dx} \Big|_0^\pi \end{aligned} \quad (14)$$

The first term vanishes if u satisfies the boundary conditions, and the second term vanishes if v satisfies the boundary conditions. Thus, L is self-adjoint.

3. Let us consider somewhat further the “momentum operator”, $p = \frac{1}{i} \frac{d}{dx}$, discussed briefly in the differential equation note. We let this operator

be an operator on the Hilbert space of square-integrable (normalizable) functions, with $x \in [a, b]$.

- (a) Find the most general boundary condition such that p is Hermitian.
 - (b) What is the domain, D_P , of p such that p is self-adjoint?
 - (c) What is the situation when $[a, b] \rightarrow [-\infty, \infty]$? Is p bounded or unbounded?
4. Prove that the different systems of orthogonal polynomials are distinguished by the weight function and the interval. That is, the system of polynomials in $[a, b]$ is uniquely determined by $w(x)$ up to a constant for each polynomial.
 5. We said that the recurrence relation for the orthogonal polynomials may be expressed in the form:

$$f_{n+1}(x) = (a_n + b_n x) f_n(x) - c_n f_{n-1}(x), \quad (15)$$

see Eqn. 16. Try to verify.

Solution: The complete statement is that the recurrence relation for the orthogonal polynomials may be expressed in the form:

$$f_{n+1}(x) = (a_n + b_n x) f_n(x) - c_n f_{n-1}(x), \quad (16)$$

where

$$b_n = \frac{k_{n+1}}{k_n} \quad (17)$$

$$a_n = b_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right) \quad (18)$$

$$c_n = \frac{h_n}{h_{n-1}} \frac{k_{n+1} k_{n-1}}{k_n^2}, \quad (19)$$

and $c_0 = 0$. The k_n notation here is defined by:

$$f_n(x) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \dots + k^{(n)}, \quad (20)$$

where $n = 0, 1, 2, \dots$. We also have the orthogonality and normalization condition:

$$\langle f_n | f_m \rangle = h_n \delta_{nm}, \quad (21)$$

where h_n is determined by the $k_n^{(\ell)}$, $\ell = 0, 1, \dots, n$ constants.

Let us write

$$p(x) = (a_n + b_n x) f_n(x) - c_n f_{n-1}(x), \quad (22)$$

where p stands for “proposed”, since we haven’t demonstrated that it is really equal to f_{n+1} yet. Since $b_n \neq 0$, the recurrence relation gives a polynomial of degree $n + 1$ for p . The scalar product of p with polynomial f_m is:

$$\langle p | f_m \rangle = a_n \langle f_n | f_m \rangle + b_n \langle x f_n | f_m \rangle - c_n \langle f_{n-1} | f_m \rangle. \quad (23)$$

Since p is a polynomial of degree $n + 1$, it is orthogonal to any f_m with $m > n + 1$, since such polynomials are orthogonal to any polynomial of lower degree by construction.

Consider $m = n + 1$:

$$\langle p | f_{n+1} \rangle = b_n \langle x f_n | f_{n+1} \rangle. \quad (24)$$

Note that:

$$f_{n+1} = k_{n+1} x^{n+1} + k'_{n+1} x^n + k''_{n+1} x^{n-1} + \dots \quad (25)$$

Thus,

$$x f_n = k_n x^{n+1} + k'_n x^n + k''_n x^{n-1} + \dots \quad (26)$$

$$= \frac{k_n}{k_{n+1}} f_{n+1} + \left(\frac{k'_n}{k_n} - \frac{k'_{n+1}}{k_{n+1}} \right) f_n + \sum_{j=0}^{n-1} \alpha_j f_j, \quad (27)$$

where the α_i are constants that we won’t need to determine explicitly, since these terms are all orthogonal to f_{n+1} and f_n . Hence,

$$\langle p | f_{n+1} \rangle = b_n \frac{k_n}{k_{n+1}} \langle f_{n+1} | f_{n+1} \rangle, \quad (28)$$

and thus $\langle p | f_{n+1} \rangle = \langle f_{n+1} | f_{n+1} \rangle$ if $b_n = k_{n+1}/k_n$.

Now consider $m = n$:

$$\langle p | f_n \rangle = a_n \langle f_n | f_n \rangle + b_n \langle x f_n | f_n \rangle \quad (29)$$

$$= a_n \langle f_n | f_n \rangle + b_n \left(\frac{k'_n}{k_n} - \frac{k'_{n+1}}{k_{n+1}} \right) \langle f_n | f_n \rangle. \quad (30)$$

This is zero if

$$a_n = b_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right). \quad (31)$$

Next, consider $m = n - 1$:

$$\langle p|f_{n-1}\rangle = b_n\langle xf_n|f_{n-1}\rangle - c_n\langle f_{n-1}|f_{n-1}\rangle \quad (32)$$

$$= b_n\langle f_n|xf_{n-1}\rangle - c_nh_{n-1}. \quad (33)$$

Noting that

$$xf_{n-1} = \frac{k_{n-1}}{k_n}f_n + \sum_{j=0}^{n-1} \beta_j f_j, \quad (34)$$

where again we needn't evaluate the β_j coefficients, we find

$$\langle p|f_{n-1}\rangle = b_n \frac{k_{n-1}}{k_n} \langle f_n|f_n\rangle - c_nh_{n-1} \quad (35)$$

$$= b_n \frac{k_{n-1}}{k_n} h_n - c_nh_{n-1}. \quad (36)$$

This is zero if

$$c_n = \frac{h_n}{h_{n-1}} \frac{k_{n+1}k_{n-1}}{k_n^2}. \quad (37)$$

It finally remains to consider $m < n - 1$

$$\langle p|f_m\rangle = b_n\langle xf_n|f_m\rangle, \quad m < n - 1, \quad (38)$$

$$= b_n\langle f_n|xf_m\rangle \quad (39)$$

$$= b_n\langle f_n|\sum_{j=0}^{m+1} \gamma_j f_j\rangle \quad (40)$$

$$= 0. \quad (41)$$

Hence, p satisfies, for the specified a_n, b_n, c_n all of the orthogonality and normalization conditions, and hence is equal to f_{n+1} because f_{n+1} is uniquely specified by these conditions. We note that the possibly special case $n = 0$ is readily checked.

6. We discussed some theorems for the qualitative behavior of classical orthogonal polynomials, and illustrated this with the one-electron atom radial wave functions. Now consider the simple harmonic oscillator (in one dimension) wave functions. The potential is

$$V(x) = \frac{1}{2}kx^2. \quad (42)$$

Thus, the Schrödinger equation is

$$-\frac{1}{2m} \frac{d^2}{dx^2} \psi(x) + \frac{1}{2}kx^2 \psi(x) = E\psi(x). \quad (43)$$

Make a sketch showing the qualitative features you expect for the wave functions corresponding to the five lowest energy levels.

Try to do this with some care: There is really a lot that you can say in qualitative terms without ever solving the Schrödinger equation. Include a curve of the potential on your graph. Try to illustrate what happens at the classical turning points (that is, the points where $E = V(x)$).

Solution:

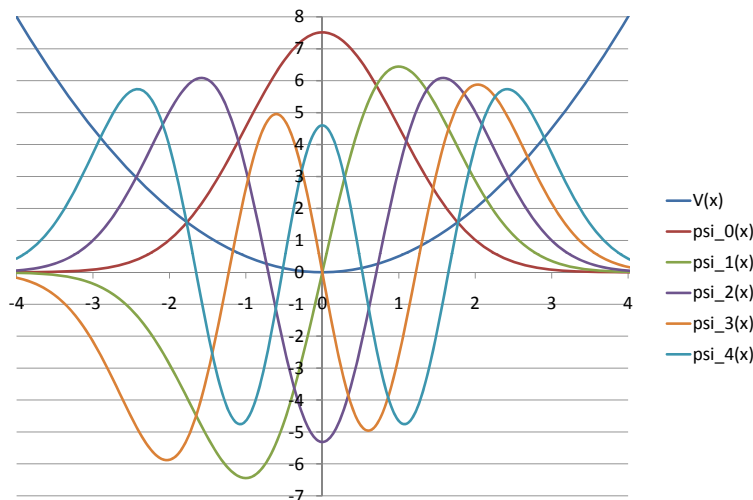


Figure 1: One dimension simple harmonic oscillator wave functions. The potential energy is $V(x) = x^2/2$, and the mass is $m = 1$. Thus the energy levels are at $n + 1/2$, and the classical turning points may be found as where these values intersect the curve for $V(x)$. The wave functions are multiplied by ten for display.

Features that a good qualitative sketch should include:

- (a) The functions and their first derivatives should be continuous everywhere.
- (b) The lowest level should have one zero, each additional level adds an additional zero.
- (c) The functions should have negative second derivative outside the classical turning points, approaching zero. The second derivative should be positive inside the classical turning points.
- (d) The zeros of successive wave functions should alternate.

- (e) The wavelength of oscillations should become shorter as the potential decreases (i.e., the second derivative increases there).
- (f) Because the potential is symmetric, the even wave functions should have even parity and the odd wave functions should have odd parity.

Note that the signs of the wave functions (in fact, their complex phases) are arbitrarily chosen.

7. Find the Green's function for the operator

$$L = \frac{d^2}{dx^2} + k^2, \quad (44)$$

where k is a constant, and with boundary conditions $u(0) = u(1) = 0$. For what values of k does your result break down? You may assume $x \in [0, 1]$.

Solution: We look for $G(x, y)$ such that $L_x G(x, y) = -\delta(x - y)$. This is in the form of a Sturm-Liouville operator with $p(x) = 1$ and $q(x) = -k^2$. Thus, we can write the Green's function as:

$$G(x, y) = A(y)u_1(x) + B(y)u_2(x) + \begin{pmatrix} - \\ + \end{pmatrix} \frac{u_1(x)u_2(y) - u_2(x)u_1(y)}{2W} \quad \begin{matrix} x \leq y \\ x \geq y \end{matrix}. \quad (45)$$

The solutions u_1 and u_2 to the homogeneous equation need not satisfy the boundary conditions. Possible choices are $u_1 = \sin kx$ and $u_2 = \cos kx$. The Wronskian is:

$$W = u_1(x)u_2'(x) - u_1'(x)u_2(x) \quad (46)$$

$$= -k. \quad (47)$$

Thus,

$$G(x, y) = A(y) \sin kx + B(y) \cos kx + \begin{pmatrix} + \\ - \end{pmatrix} \frac{\sin kx \cos ky - \cos kx \sin ky}{2k} \quad \begin{matrix} x \leq y \\ x \geq y \end{matrix}. \quad (48)$$

Now we turn our attention to the boundary conditions. The problem doesn't explicitly specify the range for x . We'll make the typical presumption that the range is implied by the limits in the boundary conditions, when not explicitly specified otherwise. Different assumptions are also acceptable.

Thus, $u(0) = 0$ gives:

$$0 = B(y) - \frac{\sin ky}{2k}, \quad (49)$$

or $B(y) = \frac{\sin ky}{2k}$. The $u(1) = 0$ condition gives:

$$0 = A(y) \sin k + \frac{\cos k \sin ky}{2k} - \frac{1}{2k}(\sin k \cos ky - \cos k \sin ky), \quad (50)$$

or

$$A(y) = \frac{\sin k \cos ky - 2 \cos k \sin ky}{2k \sin k}. \quad (51)$$

Finally, we have the Green's function:

$$G(x, y) = \frac{1}{k} \left[\theta(y - x) \sin kx \cos ky + \theta(x - y) \cos kx \sin ky - \frac{\sin kx \sin ky}{\tan k} \right]. \quad (52)$$

Note the symmetry in $x \leftrightarrow y$, as expected for this self-adjoint operator. The equation breaks down for $k = 0$, where the Wronskian vanishes. We have already dealt with the Green's function for the $k = 0$ case in the notes.

8. An integral that is encountered in calculating radiative corrections in e^+e^- collisions is of the form:

$$I(t; a, b) = \int_a^b \frac{x^{t-1}}{1-x} dx, \quad (53)$$

where $0 \leq a < b \leq 1$, and $t \geq 0$.

Show that this integral may be expressed in terms of the hypergeometric function ${}_2F_1$. Make sure to check the $t = 0$ case.

Solution: We are given the integral representation:

$${}_2F_1(\alpha, \beta; \gamma, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-yx)^{-\alpha} dx. \quad (54)$$

Thus,

$$I(t; 0, 1) = \frac{1}{t} {}_2F_1(\alpha = 1, \beta = t; \gamma = 1 + t, y = 1), \quad (55)$$

since

$$\frac{\Gamma(1+t)}{\Gamma(t)\Gamma(1)} = t. \quad (56)$$

We want the integral from a to b , so write:

$$I(t; a, b) = \int_0^b \frac{x^{t-1}}{1-x} dx - \int_0^a \frac{x^{t-1}}{1-x} dx \quad (57)$$

$$= b^t \int_0^1 \frac{x^{t-1}}{1-bx} dx - a^t \int_0^1 \frac{x^{t-1}}{1-ax} dx \quad (58)$$

$$= \frac{b^t}{t} {}_2F_1(1, t; 1+t, b) - \frac{a^t}{t} {}_2F_1(1, t; 1+t, a). \quad (59)$$

What about the $t = 0$ case? Note that ${}_2F_1(1, t = 0; \gamma, y)$ is not really defined. However, we may be able to make sense out of our expression anyway. Consider the series expansion:

$${}_2F_1(\alpha, \beta; \gamma, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{y^n}{n!}. \quad (60)$$

Thus,

$$I(t; a, b) = \frac{\Gamma(1+t)}{\Gamma(t)\Gamma(1)} \frac{1}{t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma(n+t)}{\Gamma(n+t+1)} \left(\frac{b^{n+t}}{n!} - \frac{a^{n+t}}{n!} \right) \quad (61)$$

$$= \sum_{n=0}^{\infty} \frac{b^{n+t} - a^{n+t}}{n+t}. \quad (62)$$

Taking the $t \rightarrow 0$ limit:

$$\lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{b^{n+t} - a^{n+t}}{n+t} = \lim_{t \rightarrow 0} \frac{b^t - a^t}{t} + \sum_{n=1}^{\infty} \frac{b^n - a^n}{n} \quad (63)$$

$$= \lim_{t \rightarrow 0} \frac{e^{t \ln b} - e^{t \ln a}}{t} + \ln \frac{1-a}{1-b} \quad (64)$$

$$= \lim_{t \rightarrow 0} \frac{1 + t \ln b - 1 - t \ln a}{t} + \ln \frac{1-a}{1-b} \quad (65)$$

$$= \ln \frac{b(1-a)}{a(1-b)}. \quad (66)$$

This may be compared with

$$I(0; a, b) = \int_a^b \frac{dx}{x(1-x)} \quad (67)$$

$$= \int_a^b \left(\frac{1}{x} + \frac{1}{1-x} \right) dx \quad (68)$$

$$= \ln \frac{b(1-a)}{a(1-b)}. \quad (69)$$

As long as $a > 0$ and $b < 1$ we obtain the correct answer even for $t = 0$.

9. We consider the Helmholtz equation in three dimensions:

$$\nabla^2 u + k^2 u = 0 \quad (70)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Such a situation may arise, for example, if we are interested in the electric field inside a conducting sphere. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (71)$$

with $G(r = a, \mathbf{y}) = 0$. We'll do this via one approach in this problem, and try another approach in the next problem.

Find $G(\mathbf{x}, \mathbf{y})$ by obtaining solutions to the homogeneous equation

$$(\nabla^2 + k^2)G = 0, \quad (72)$$

on either side of $r = |\mathbf{y}|$; satisfying the boundary conditions at $r = a$, and the appropriate matching conditions at $r = |\mathbf{y}|$.

Solution: The symmetry of the problem is such that it is convenient to work in spherical polar coordinates.

Let $r = |\mathbf{x}|$ and $r' = |\mathbf{y}|$. Let $r_<$ be the smaller of (r, r') and let $r_>$ be the larger of (r, r') . Let (θ, ϕ) be the polar and azimuthal angles of \mathbf{x} , and likewise let (θ', ϕ') be the polar and azimuthal angles of \mathbf{y} . Let ψ be the angle between \mathbf{x} and \mathbf{y} , that is,

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad (73)$$

The final answer is:

$$G(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \frac{k(2\ell + 1)P_{\ell}(\cos \psi)}{4\pi} \left\{ \frac{j_{\ell}(kr_<)}{j_{\ell}(ka)} [j_{\ell}(ka)n_{\ell}(kr_>) - n_{\ell}(ka)j_{\ell}(kr_>)] \right\}. \quad (74)$$

10. We return to the preceding problem. This is the problem of the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \quad (75)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Such a situation may arise, for example, if we are interested in the electric field inside a conducting sphere. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (76)$$

with $G(r = a, \mathbf{y}) = 0$.

In problem 9, you found $G(\mathbf{x}, \mathbf{y})$ by obtaining solutions to the homogeneous equation

$$(\nabla^2 + k^2)G = 0, \quad (77)$$

on either side of $r = |\mathbf{y}|$; satisfying the boundary conditions at $r = a$, and the appropriate matching conditions at $r = |\mathbf{y}|$.

Now we take a different approach: Find G by directly solving $(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$. You should ignore the boundary conditions at first and obtain a solution by integrating the equation over a small volume containing \mathbf{y} . Then satisfy the boundary conditions by adding a suitable function $g(\mathbf{x}, \mathbf{y})$ that satisfies $(\nabla_x^2 + k^2)g(\mathbf{x}, \mathbf{y}) = 0$ everywhere.

11. Let's continue our discussion of the preceding two problems. This is the problem of the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \quad (78)$$

inside a sphere of radius a , subject to the boundary condition $u(r = a) = 0$. Our goal is to find $G(\mathbf{x}, \mathbf{y})$ such that

$$(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad (79)$$

with $G(r = a, \mathbf{y}) = 0$.

In problem 10, you found $G(\mathbf{x}, \mathbf{y})$ by directly solving $(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, ignoring the boundary conditions at first. This is called the "fundamental solution" because it contains the desired singularity structure, and hence has to do with the "source". Now find the fundamental solution by another technique: Put the origin at \mathbf{y} and solve the equation

$$(\nabla_x^2 + k^2)f(\mathbf{x}) = \delta(\mathbf{x}), \quad (80)$$

by using Fourier transforms. Do you get the same answer as last week?

12. Referring still to the Helmholtz problem (problems 10 – 11), discuss the relative merits of the solutions found in problems 9 and 10. In particular, analyze, by making a suitable expansion, a case where the problem 10 solution is likely to be preferred, stating the necessary assumptions clearly.
13. We noted that the Green's function method is applicable beyond the Sturm-Liouville problem. For example, consider the differential operator:

$$L = \frac{d^4}{dx^4} + \frac{d^2}{dx^2}. \quad (81)$$

As usual, we wish to find the solution to $Lu = -\phi$. Let us consider the case of boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$.

- (a) Find the Green's function for this operator.
- (b) Find the solution for $x \in [0, \infty]$ and $\phi(x) = e^{-x}$.

You are encouraged to notice, at least in hindsight, that you could probably have solved this problem by elementary means.

14. Using the Green's function method, we derived in class the time development transformation for the free-particle Schrödinger equation in one dimension:

$$U(x, y; t) = \frac{1}{\sqrt{2}} \left(1 - i \frac{t}{|t|}\right) \sqrt{\frac{m}{2\pi|t|}} \exp\left[\frac{im(x-y)^2}{2t}\right]. \quad (82)$$

This should have the property that if you do a transformation by time t , followed by a transformation by time $-t$, you should get back to where you started. Check whether this is indeed the case or not.

15. Using the Christoffel-Darboux formula, find the projection operator onto the subspace spanned by the first three Chebyshev polynomials.
16. We discussed the radial solutions to the “one-electron” Schrödinger equation. Investigate orthogonality of the result – are our wave functions orthogonal or not?
17. In class we considered the problem with the Hamiltonian

$$H = -\frac{1}{2m} \frac{d^2}{dx^2}. \quad (83)$$

Let us modify the problem somewhat and consider the configuration space $x \in [a, b]$ (“infinite square well”).

- (a) Construct the Green's function, $G(x, y; z)$ for this problem.
- (b) From your answer to part (a), determine the spectrum of H .
- (c) Notice that, using

$$G(x, y; z) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k^*(y)}{\omega_k - z}, \quad (84)$$

the normalized eigenstate, $\phi_k(x)$, can be obtained by evaluating the residue of G at the pole $z = \omega_k$. Do this calculation, and check that your result is properly normalized.

- (d) Consider the limit $a \rightarrow -\infty$, $b \rightarrow \infty$. Show, in this limit that $G(x, y; z)$ tends to the Green's function we obtained in class for this Hamiltonian on $x \in (-\infty, \infty)$:

$$G(x, y; z) = i\sqrt{\frac{m}{2z}}e^{i\rho|x-y|}. \quad (85)$$

18. Let us investigate the Green's function for a slightly more complicated situation. Consider the potential:

$$V(x) = \begin{cases} V & |x| \leq \Delta \\ 0 & |x| > \Delta \end{cases} \quad (86)$$

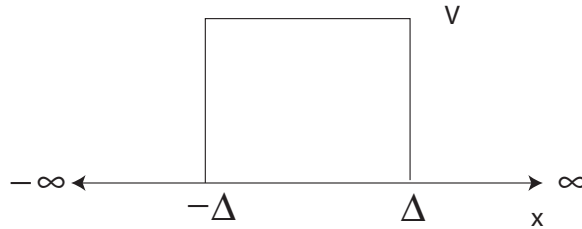


Figure 2: The “finite square potential”.

- (a) Determine the Green's function for a particle of mass m in this potential.

Remarks: You will need to construct your “left” and “right” solutions by considering the three different regions of the potential, matching the functions and their first derivatives at the boundaries. Note that the “right” solution may be very simply obtained from the “left” solution by the symmetry of the problem. In your solution, let

$$\rho = \sqrt{2m(z - V)} \quad (87)$$

$$\rho_0 = \sqrt{2mz}. \quad (88)$$

Make sure that you describe any cuts in the complex plane, and your selected branch. You may find it convenient to express your answer to some extent in terms of the force-free Green's function:

$$G_0(x, y; z) = \frac{im}{\rho}e^{i\rho_0|x-y|}. \quad (89)$$

- (b) Assume $V > 0$. Show that your Green's function $G(x, y; z)$ is analytic in your cut plane, with a branch point at $z = 0$.
- (c) Assume $V < 0$. Show that $G(x, y; z)$ is analytic in your cut plane, except for a finite number of simple poles at the bound states of the Hamiltonian.
19. In class, we obtained the free particle propagator for the Schrödinger equation in quantum mechanics:

$$U(x, t; x_0, t_0) = \frac{1}{\sqrt{2}} \left(1 - i \frac{t - t_0}{|t - t_0|} \right) \sqrt{\frac{m}{2\pi|t - t_0|}} \exp \left[\frac{im(x - x_0)^2}{2(t - t_0)} \right]. \quad (90)$$

Let's actually use this to evolve a wave function. Thus, let the wave function at time $t = t_0 = 0$ be:

$$\psi(x_0, t_0 = 0) = \left(\frac{1}{\pi a^2} \right)^{1/4} \exp \left(-\frac{x_0^2}{2a^2} + ip_0 x_0 \right), \quad (91)$$

where a and p_0 are real constants. Since the absolute square of the wave function gives the probability, this wave function corresponds to a Gaussian probability distribution (i.e., the probability density function to find the particle at x_0) at $t = t_0$:

$$|\psi(x_0, t_0)|^2 = \left(\frac{1}{\pi a^2} \right)^{1/2} e^{-\frac{x_0^2}{a^2}}. \quad (92)$$

The standard deviation of this distribution is $\sigma = a/\sqrt{2}$. Find the probability density function, $|\psi(x, t)|^2$, to find the particle at x at some later (or earlier) time t . You are encouraged to think about the "physical" interpretation of your result.