

**Physics 129a**  
**Introduction to Distributions**  
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## 1 Motivation

We are often faced with the following problem: Given a linear operator  $L$  and a (Hilbert) space  $V$ , we wish to solve for  $x$  in:

$$Lx = a, \tag{1}$$

where  $a, x \in V$ , and  $a$  is a given vector. We here think in terms of a Hilbert space, though that is not necessarily required as long as the vector space is sufficiently large for solutions to exist, etc. For example, we may be trying to solve an integral equation  $f = g + \lambda Kf$ , where  $L = 1 - \lambda K$ ,  $a = g$ , and  $x = f$ .

We can identify (at least) two approaches to a solution:

1. Find the eigenvalues and eigenvectors of  $L$ :

$$L\beta = \lambda\beta, \tag{2}$$

where  $\lambda$  is a constant. Then, if the eigenvectors form a “complete” set, we find a solution to  $Lx = a$  by expanding  $a$  and  $x$  in terms of the (orthonormalized) eigenvectors:

$$\begin{aligned} a &= \sum_i a_i \beta_i & a_i &= \langle \beta_i | a \rangle \\ x &= \sum_i x_i \beta_i & x_i &= \langle \beta_i | x \rangle, \end{aligned}$$

where  $L\beta_i = \lambda_i \beta_i$ .

Then  $Lx = a$  implies

$$L \sum_i x_i \beta_i = \sum_i \lambda_i x_i \beta_i = \sum_i a_i \beta_i. \tag{3}$$

With  $\langle \beta_i | \beta_j \rangle = \delta_{ij}$ , we can solve for the unknown expansion coefficients  $x_i$ :

$$x_i = a_i / \lambda_i. \tag{4}$$

The solution to the problem is thus:

$$\begin{aligned} |x\rangle &= \sum_i |\beta_i\rangle \frac{\langle \beta_i | a \rangle}{\lambda_i} \\ &= \left( \sum_i \frac{|\beta_i\rangle \langle \beta_i|}{\lambda_i} \right) |a\rangle. \end{aligned} \tag{5}$$

The quantity in parentheses is called an **infinite-term dyad** (or an  $n$ -term dyad if the sum is over a finite number of  $n$  terms).

2. Find the inverse of  $L$ , in which case the solution to the equation is

$$x = L^{-1}a. \quad (6)$$

Note that we effectively accomplished this in our first approach, with

$$L^{-1} = \sum_i \frac{|\beta_i\rangle\langle\beta_i|}{\lambda_i}, \quad (7)$$

so that

$$LL^{-1} = \sum_i L \frac{|\beta_i\rangle\langle\beta_i|}{\lambda_i} = \sum_i |\beta_i\rangle\langle\beta_i|. \quad (8)$$

The final expression above is the identity operator if the eigenfunctions are complete. Note that we may also write the expansion of the operator in terms of the eigenvectors:

$$L = \sum_i \lambda_i |\beta_i\rangle\langle\beta_i|. \quad (9)$$

Let us turn to something more specific. We know how to carry out the program we have just outlined in the case of matrix equations, and for  $L = I + K$ , where  $K$  is an integral operator. Now we are interested in the more subtle case where  $L$  is a linear differential operator. Rewrite the problem as  $Lf = a$ , where  $f = f(x)$  and  $a = a(x)$ . Taking the second approach above, we might guess that  $L^{-1}$  is an integral operator:

$$f(x) = \int G(x, y)a(y)dy, \quad (10)$$

and that the problem is one of finding  $G(x, y)$ .

However, we encounter a problem in carrying out this approach. Suppose  $L^{-1}$  is this integral operator. Let  $c(x)$  be any continuous function. Then

$$\begin{aligned} c(x) &= (LL^{-1}c)(x) \\ &= L_x \int G(x, y)c(y)dy \\ &= \int L_x G(x, y)c(y)dy, \end{aligned} \quad (11)$$

where  $L_x$  is the differential operator  $L$ , with respect to variable  $x$ . Define  $\delta(x, y) \equiv L_x G(x, y)$ . Then

$$c(x) = \int \delta(x, y)c(y)dy. \quad (12)$$

The only way we can satisfy Eqn. 12 for arbitrary continuous functions  $c$  is for  $\delta(x, y) = 0$  whenever  $x \neq y$ . That is, we must have

$$\delta(x, y) = \delta(x - y). \quad (13)$$

Then

$$c(x) = \int \delta(x - y)c(y)dy. \quad (14)$$

In particular, for  $c(x) = 1$ , we have

$$1 = \int \delta(x - y)dy. \quad (15)$$

But we cannot interpret this  $\delta(x)$  “function” as a vector in our Hilbert space of functions, since  $\delta(x - y) \neq 0$  at only a single point (a set of measure zero here). Consider the difficulty of “normalizing” such a vector, i.e., what is

$$\int \delta^2(x)dx? \quad (16)$$

In spite of this difficulty, we are well-aware that this is a useful approach in solving practical physics problems. How can we interpret  $\delta(x)$  in a mathematically meaningful manner?

## 2 Distributions

**Definition (Distribution):** A **distribution** (or **generalized function**) is a continuous, linear functional defined on some set of **test functions** (of one real variable).

Thus, a distribution is a linear operator, with domain given by some suitable subset of the space of functions of a real variable, and range in the space of complex numbers. A distribution is a mapping of functions to complex numbers, or, in other words of vectors to scalars. For simplicity, we will deal with real functions and real numbers in this discussion.

Note that the “ $\delta$ -function”, applied as an integral operator, appears to have the desired properties:

$$F_\delta(f) = \int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \quad (17)$$

$$\begin{aligned} F_\delta(c_1f_1 + c_2f_2) &= \int_{-\infty}^{\infty} \delta(x) [c_1f_1(x) + c_2f_2(x)] dx \\ &= c_1f_1(0) + c_2f_2(0) \\ &= c_1F_\delta(f_1) + c_2F_\delta(f_2). \end{aligned} \quad (18)$$

However, we still need to address the question of the set of “test functions”. For our purposes, especially given that we are interested in differential equations, a suitable set of test functions is the following “nice set”:

Our set of test functions, which shall be denoted  $\mathcal{T}$ , is the set of all continuous infinitely differentiable functions with bounded support (or, “compact support”).

To be clear, the **support** of a function  $f(x)$  is the closure of the set on which  $f(x) \neq 0$ . Thus, here bounded support means that the function must vanish identically outside of some finite interval. For example, the following function belongs to  $\mathcal{T}$ :

$$\tau(x; a) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & |x| < a, \\ 0 & |x| \geq a \end{cases} \tag{19}$$

$$\tau(0; a) = \frac{1}{e}, \tag{20}$$

where  $a$  is a positive real number.

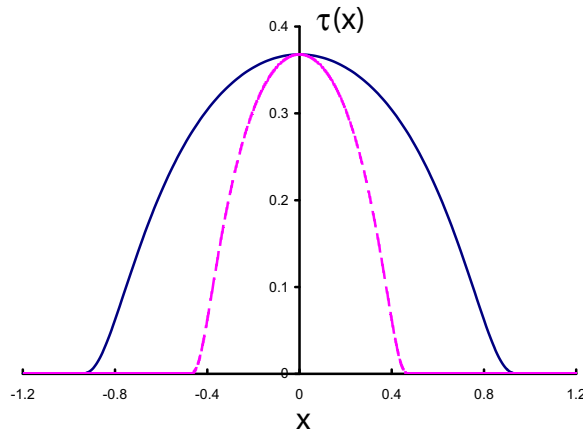


Figure 1: Illustration of the function  $\tau(x; a)$ , for  $a = 1$  (solid curve) and  $a = 0.5$  (dashed curve).

The reader may check that the space  $\mathcal{T}$  is a linear space.

Our class of test functions is rather restrictive. However, it is sufficient for us according to the following theorem:

**Theorem: Approximation Theorem:** Given any  $\epsilon > 0$ , any continuous function  $f(x)$  with bounded support may be uniformly approximated within  $\epsilon$  by some function  $t(x) \in \mathcal{T}$ . Further, the support of  $t$  may be contained within an arbitrary neighborhood of the support of  $f$ .

The phrase “within  $\epsilon$ ” means that there exists  $t \in \mathcal{T}$  such that  $|f(x) - t(x)| < \epsilon$  for all  $x$ . The approximation of the support means that we can insure that the function  $t$  doesn’t manage to “pop-up” (within  $\epsilon$ ) somewhere outside the support of  $f$ . Proof of this theorem is left to the reader.

Now we make our notions a bit more precise:

**Definition:** We define  $F(t)$  to be a **linear functional** if, given any scalars  $c_1, c_2$  and any  $t_1, t_2 \in \mathcal{T}$ :

$$F(c_1 t_1 + c_2 t_2) = c_1 F(t_1) + c_2 F(t_2), \quad (21)$$

and  $F(t)$  is a **continuous functional** if, given any sequence of test functions  $t_1(x), t_2(x), \dots$  which converges to zero in  $\mathcal{T}$ , then  $F(t_1), F(t_2), \dots$  converges to zero.

That is, if we pick any sequence of test functions  $t_1, t_2, \dots$  such that all  $t_n(x)$  vanish identically outside of a common interval and such that

$$\lim_{n \rightarrow \infty} \frac{d^m t_n(x)}{dx^m} = 0, \quad m = 0, 1, 2, \dots, \quad (22)$$

then  $F$  is continuous if

$$\lim_{n \rightarrow \infty} F(t_n) = 0. \quad (23)$$

We note that  $F_\delta(t) \equiv t(0)$  defines a continuous linear functional.

Other notations may be employed. Considering  $F_\delta$ , we could also write the suggestive forms:

$$F_\delta(t) = (\delta, t) = \int_{-\infty}^{\infty} \delta(x)t(x)dx \equiv t(0). \quad (24)$$

Generalization to higher dimensions is also possible:

$$F_\delta(t) = (\delta(\mathbf{x}), t(\mathbf{x})) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta(\mathbf{x})t(\mathbf{x})d^{(n)}(\mathbf{x}) = t(\mathbf{0}). \quad (25)$$

Here, we could also write  $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2) \dots \delta(x_n)$ .

We may define a vector space of continuous linear functionals: Let  $\mathcal{F}$  be the set of all continuous linear functionals defined on  $\mathcal{T}$ . Let  $f \in \mathcal{F}$ ,  $t \in \mathcal{T}$ , and  $\alpha$  be a scalar. Define  $(\alpha f, t) = F_{\alpha f}(t) \equiv (f, \alpha t)$ . Then

$$(\alpha f, t) = \alpha(f, t) = \alpha F_f(t). \quad (26)$$

Further define, for  $f_1, f_2 \in \mathcal{F}$ :

$$(f_1 + f_2, t) \equiv (f_1, t) + (f_2, t). \quad (27)$$

Thus,  $\mathcal{F}$  is a vector space of continuous linear functionals on  $\mathcal{T}$ .

We also may define a concept of convergence in our vector space  $\mathcal{F}$ : The sequence  $f_1, f_2, \dots \in \mathcal{F}$  is said to converge to  $f \in \mathcal{F}$  if for any  $t \in \mathcal{T}$ :

$$\lim_{n \rightarrow \infty} (f_n, t) = (f, t). \quad (28)$$

Note that it is a theorem that  $\mathcal{F}$  is **complete**. That is, every sequence  $f_1, f_2, \dots \in \mathcal{F}$  for which  $\lim_{n \rightarrow \infty} (f_n, t)$  converges for any  $t \in \mathcal{T}$  converges to an element of  $\mathcal{F}$ .

### 3 Practical Matters

We turn now to some “practical” matters in the use of generalized functions.

First, consider the multiplication by an infinitely-differentiable function. If  $a(x)$  is an infinitely-differentiable function, then we generalize our earlier definition for multiplication by a scalar to obtain:

$$F_{af}(t) = (af, t) = (f, at). \quad (29)$$

Note that if  $t \in \mathcal{T}$  then  $at \in \mathcal{T}$ , so this is well-defined. For example,

$$(x\delta(x), t(x)) = (\delta(x), xt(x)) = 0 \cdot t(0) = 0. \quad (30)$$

Second, consider a mapping of the underlying space. Suppose  $M$  defines a  $1 : 1$  mapping (not necessarily a linear one) of the space to which  $\mathbf{x}$  belongs to itself:

$$M\mathbf{x} = \mathbf{x}'. \quad (31)$$

For an arbitrary function  $g(\mathbf{x})$  this mapping generates a new function:

$$g'(\mathbf{x}) = (Mg)(\mathbf{x}) = g(M^{-1}\mathbf{x}). \quad (32)$$

Note that this definition implies  $g'(\mathbf{x}') = g(\mathbf{x})$ , that is, the value of the new function, evaluated at the same space point, is unchanged. We often call  $g'$  and  $g$  by the same name, but it is important to remember that they are really different functional forms of their arguments.

We may apply this same procedure to generalized functions as well as to ordinary ones. For example, suppose  $\mathbf{x} \in R^n$  and consider

$$(f(M^{-1}\mathbf{x}), t(\mathbf{x})) = \int_{(\infty)} f(M^{-1}\mathbf{x})t(\mathbf{x})d^n(\mathbf{x}). \quad (33)$$

Suppose for the time being that  $f$  is an ordinary function. With the above, it clearly also defines a generalized function. Then we have

$$(f(M^{-1}\mathbf{x}), t(\mathbf{x})) = \int_{(\infty)} f(\mathbf{y})t(M\mathbf{y})|J|d^n(\mathbf{y}), \quad (34)$$

where

$$J = \det \left( \frac{\partial \mathbf{x}}{\partial \mathbf{y}} \right) \quad (35)$$

is the Jacobian of the transformation between  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\mathbf{x} = M\mathbf{y}$ .

Thus, if  $|J|t(M\mathbf{x}) \in \mathcal{T}$ , we find:

$$((Mf)(\mathbf{x}), t(\mathbf{x})) = (f(M^{-1}\mathbf{x}), t(\mathbf{x})) = (f(\mathbf{x}), |J|t(M\mathbf{x})), \quad (36)$$

and this result may be extended to arbitrary distributions  $f \in \mathcal{F}$ .

This result leads to one of the most handy equations in using delta functions. Suppose  $f(x)$  is a monotonic 1 : 1 mapping of  $R^1 \rightarrow R^1$ , with  $f(a) = 0$ . Then

$$\begin{aligned}
 (\delta[f(x)], t(x)) &= \int_{-\infty}^{\infty} \delta[f(x)] t(x) dx \\
 &= \int_{-\infty}^{\infty} \delta(y) t[f^{-1}(y)] \frac{dy}{|f'[f^{-1}(y)]|} \\
 &= \frac{t[f^{-1}(0)]}{|f'[f^{-1}(0)]|} \\
 &= \frac{t(a)}{|f'(a)|}.
 \end{aligned} \tag{37}$$

Alternatively, we could write

$$\delta[f(x)] = \frac{\delta(x - a)}{|f'(a)|}. \tag{38}$$

Note that this implies the symmetry  $\delta(-x) = \delta(x)$ .

This result may be generalized to functions which have multiple zeros. Suppose  $f$  has zeros at  $a_1, a_2, \dots$ . Then

$$\delta[f(x)] = \sum_i \frac{\delta(x - a_i)}{|f'(a_i)|}. \tag{39}$$

Let us now consider differentiation of distributions. We may generalize from the fundamental theorem for ordinary functions:

$$\begin{aligned}
 (f', t) &= \int_{-\infty}^{\infty} f'(x) t(x) dx \\
 &= f(x) t(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) t'(x) dx \\
 &= -(f, t').
 \end{aligned} \tag{40}$$

Here, the integrated part is zero because  $t$  has bounded support, and  $t'$  exists because  $t$  is infinitely differentiable.

To obtain the  $n$ -th derivative, we simply repeat the argument, obtaining

$$(f^{(n)}, t) = (-1)^n (f, t^{(n)}). \tag{41}$$

For the case of the delta-functional,

$$(\delta', t) = -t'(0), \tag{42}$$

$$(\delta'', t) = t''(0). \tag{43}$$

The reader is encouraged to draw some pictures, with a “delta-function” of small but finite width, to see that these expressions match intuition.

For another example, consider the “Heavyside unit function(al)”:

$$(\theta, t) \equiv \int_{-\infty}^{\infty} \theta(x)t(x)dx, \quad (44)$$

where

$$\theta(x) = \begin{cases} 1 & x > 0, \\ 1/2 & x = 0, \\ 0 & x < 0. \end{cases} \quad (45)$$

Then

$$\begin{aligned} (\theta', t) &= -(\theta, t') \\ &= -\int_0^{\infty} t'(x)dx \\ &= t(0), \end{aligned} \quad (46)$$

since  $t(\infty) = 0$ . Therefore,  $\theta' = \delta$ , the derivative of the step functional is the delta functional.

We remark that it is possible to rigorously define a notion of support of a generalized function, and if a generalized function has bounded support, then the conditions of bounded support and infinite smoothness of the test functions may be relaxed. Since  $\delta(x)$  has a one-point support, it may be applied to any function, as long as that function is finite at  $x = 0$ . For example, consider  $g(x) = 1 \notin \mathcal{T}$ . But there is nothing wrong with

$$(\delta, g) = \int_{-\infty}^{\infty} \delta(x)dx = 1. \quad (47)$$

Also, we may consider finite intervals:

$$\int_a^b \delta(x)f(x)dx = \begin{cases} 0 & \text{if } 0 \notin [a, b], \\ f(0) & \text{if } 0 \in (a, b). \end{cases} \quad (48)$$

As long as  $a \neq 0$  and  $b \neq 0$ , this is fine.

Finally, it is interesting to ask whether we can extend the notion of integration to generalized functions. Consider

$$\left( \int_{-\infty}^x f(y)dy, t(x) \right) = \int_{-\infty}^{\infty} \int_{-\infty}^x f(y)dyt(x)dx \quad (49)$$

$$= \left[ \int_{-\infty}^x f(y)dy \right] \left[ -\int_x^{\infty} t(y)dy \right] \Big|_{-\infty}^{\infty} \quad (50)$$

$$+ \int_{-\infty}^{\infty} f(x) \int_x^{\infty} t(y)dydx,$$

where the integrated part vanishes. That is,

$$\left( \int_{-\infty}^x f(y)dy, t(x) \right) = \left( f(x), \int_x^{\infty} t(y)dy \right). \quad (51)$$

However, we must be careful here, because in general  $\int_x^\infty t(y)dy$  does not have bounded support and hence is not in  $\mathcal{T}$ . Thus, we cannot define generally integrals of distributions. On the other hand, we often can, for example the integral of the  $\delta$ -functional is the  $\theta$ -functional:

$$\int_{-\infty}^x \delta(y)dy = \theta(x). \quad (52)$$

## 4 Exercises

1. In class we tried an example of integrating over a “delta function” involving two-body Lorentz Invariant Phase Space (LIPS). Let us try it for three body LIPS. Thus, suppose we start with an initial energy  $E$  and three-momentum  $\mathbf{p} = \mathbf{0}$ . We write the phase space volume element as:

$$dLIPS = (2\pi)^4 \delta^{(4)}(p - p_1 - p_2 - p_3) \frac{d^3(\mathbf{p}_1)}{(2\pi)^3 2E_1} \frac{d^3(\mathbf{p}_2)}{(2\pi)^3 2E_2} \frac{d^3(\mathbf{p}_3)}{(2\pi)^3 2E_3}, \quad (53)$$

where  $p, p_1, p_2, p_3$  denote four-vectors. Integrate this differential over  $d^3(\mathbf{p}_3)$  to get a differential of the form:

$$dLIPS_{12} = \int \frac{dLIPS}{d^3(\mathbf{p}_3)} d^3(\mathbf{p}_3). \quad (54)$$

This is a differential in  $d^3(\mathbf{p}_1)d^3(\mathbf{p}_2)$ . Assume that you may pick a coordinate system orientation arbitrarily (that is, there is no preferred direction specified in the initial state). With this symmetry, integrate over the angular variables of particle two. You should have remaining a differential quantity in  $d^3(\mathbf{p}_1)d|\mathbf{p}_2|$ , where the angular variables of particle 1 are measured with respect to the particle 2 direction. Assume that there is no special azimuthal direction determined by particle 2, and integrate over the azimuth angle  $\phi_1$ , of particle 1.

Finally, integrate over  $\theta_1$ , the polar angle of particle 1 (the angle between particle 1 and particle 2). Express your result as a differential in  $E_1$  and  $E_2$ .

Note that the only “physics” you need for this problem is the relation between the rest mass, energy and magnitude of three momentum of a particle:

$$E_i^2 = m_i^2 + |\mathbf{p}_i|^2. \quad (55)$$

2. In this note we considered the set of test functions:

$$\tau(x; a) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & |x| < a, \\ 0 & |x| > a \end{cases} \quad (56)$$

where  $a$  is a positive real number. Consider making a Taylor series expansion about  $x = a$ . What happens? In spite of the “niceness” of this function, you should encounter a problem. To understand this, show that, as a function of the complex variable  $z$ ,  $\tau(z; a)$  is not analytic at  $z = a$ . What is the nature of the singularity at  $z = a$  (removable?, simple?, order  $n$ ?, essential?, isolated?, not isolated?)?

3. Prove that

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x), \quad (57)$$

where  $f(x)$  is an infinitely differentiable function. Convince yourself that it makes “intuitive” sense.

4. Consider the differential equation

$$u''(x) = a(x), \quad (58)$$

where  $x \in (-\infty, \infty)$  and  $a(x)$  is some known function of  $x$ . By considering solutions to

$$\frac{d^2G(x, y)}{dx^2} = \delta(x - y), \quad (59)$$

do the following:

(a) Solve for  $u(x)$  subject to the boundary conditions:

$$u(0) = u'(0) = 0. \quad (60)$$

(b) Solve for  $u(x)$  subject to the boundary conditions:

$$u(0) = \int_0^1 u(x)dx = 0. \quad (61)$$

(c) Can  $u''(x) = a(x)$  be solved subject to the boundary conditions:

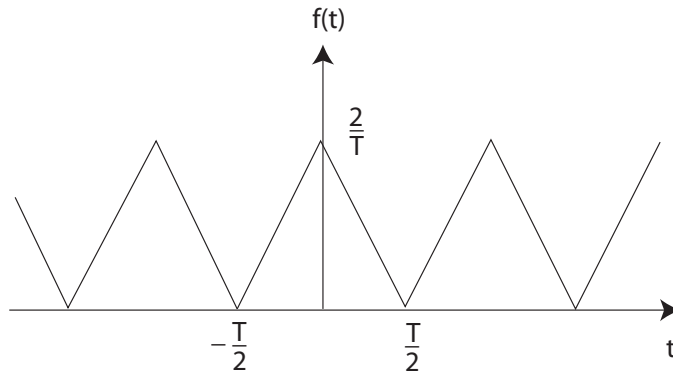
$$u(0) = u(1), \quad \text{and} \quad u'(0) = u'(1)? \quad (62)$$

5. Let us combine some of the things we have learned, in distribution theory and integral transforms, and see how the Fourier series may be interpreted as a special case of the Fourier transform. This has connections with sampling theory and the discrete Fourier transform.

We will consider here the example of a triangular “waveform”:

$$f(t) = \frac{4}{T^2} \left( \frac{T}{2} - |\text{mod}(t + T/2, T) - T/2| \right), \quad (63)$$

where  $f(t)$  is periodic with period  $T$ , see the figure.



- (a) Find the Fourier series expansion for this waveform.
- (b) Let  $d(t)$  be a distribution obtained by an infinite sequence of  $\delta$ -functionals:

$$d(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \quad (64)$$

This is periodic, with period chosen to be the same as our triangular waveform. As a periodic “function”, we can hope that it can be expanded in a Fourier series. Using this approach, or any other approach you wish, find the Fourier transform,  $\hat{d}(\omega)$ , of  $d(t)$ . In principle, we need to justify the manipulations of the distribution, but you aren’t required to here.

- (c) Now let us investigate using the Fourier integral to obtain the result of part (a). Let  $h(t)$  be a single period of the triangular waveform:

$$h(t) = \begin{cases} f(t) & |t| < T/2, \\ 0 & \text{otherwise.} \end{cases} \quad (65)$$

Express the periodic waveform  $f(t)$  as the convolution of  $h(t)$  with an infinite sequence of “impulses” ( $\delta$ -functionals) evenly spaced at intervals of  $T$ .

- (d) Using the Fourier transforms of  $h(t)$  and of the infinite sequence of  $\delta$ 's, plus the convolution theorem, determine the Fourier transform of  $f(t)$ . Compare with the result of part (a).