

Physics 129a
Introduction to Distributions
Solutions to Problems
051015 Frank Porter
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1 Exercises

1. In class we tried an example of integrating over a “delta function” involving two-body Lorentz Invariant Phase Space (LIPS). Let us try it for three body LIPS. Thus, suppose we start with an initial energy E and three-momentum $\mathbf{p} = \mathbf{0}$. We write the phase space volume element as:

$$dLIPS = (2\pi)^4 \delta^{(4)}(p - p_1 - p_2 - p_3) \frac{d^3(\mathbf{p}_1)}{(2\pi)^3 2E_1} \frac{d^3(\mathbf{p}_2)}{(2\pi)^3 2E_2} \frac{d^3(\mathbf{p}_3)}{(2\pi)^3 2E_3}, \quad (1)$$

where p, p_1, p_2, p_3 denote four-vectors. Integrate this differential over $d^3(\mathbf{p}_3)$ to get a differential of the form:

$$dLIPS_{12} = \int \frac{dLIPS}{d^3(\mathbf{p}_3)} d^3(\mathbf{p}_3). \quad (2)$$

Solution: This part is easy:

$$dLIPS_{12} = \int \frac{dLIPS}{d^3(\mathbf{p}_3)} d^3(\mathbf{p}_3) \quad (3)$$

$$= \frac{1}{8(2\pi)^5} \delta(E - E_1 - E_2 - E_3) \frac{d^3(\mathbf{p}_1) d^3(\mathbf{p}_2)}{E_1 E_2 E_3}, \quad (4)$$

where now $E_3 = \sqrt{m_3^2 + (\mathbf{p}_1 + \mathbf{p}_2)^2}$. On to the next piece...

This is a differential in $d^3(\mathbf{p}_1) d^3(\mathbf{p}_2)$. Assume that you may pick a coordinate system orientation arbitrarily (that is, there is no preferred direction specified in the initial state). With this symmetry, integrate over the angular variables of particle two.

Solution: Write $d^3(\mathbf{p}_2) = |\mathbf{p}_2|^2 d|\mathbf{p}_2| d\cos\theta_2 d\phi_2$. The fact that we can pick our coordinate orientation arbitrarily means that the physics must be independent of the direction of particle two. Thus, we integrate over $\cos\theta_2$ and ϕ_2 to get a factor of 4π :

$$\int \frac{dLIPS_{12}}{d\cos\theta_2 d\phi_2} d\cos\theta_2 d\phi_2 = \frac{1}{4(2\pi)^4} \delta(E - E_1 - E_2 - E_3) \frac{d^3(\mathbf{p}_1) |\mathbf{p}_2|^2 d|\mathbf{p}_2|}{E_1 E_2 E_3}, \quad (5)$$

Next part...

You should have remaining a differential quantity in $d^3(\mathbf{p}_1)d|\mathbf{p}_2|$, where the angular variables of particle 1 are measured with respect to the particle 2 direction. Assume that there is no special azimuthal direction determined by particle 2, and integrate over the azimuth angle ϕ_1 , of particle 1.

Solution: Again, we are using the assumption that the initial state picks out no special directions to argue that the physics is independent of the azimuth angle of particle one in the coordinate system defined by particle 2. Write $d^3(\mathbf{p}_1) = |\mathbf{p}_1|^2 d|\mathbf{p}_1| d\cos\theta_1 d\phi_1$. Integrate over ϕ to get a factor of 2π :

$$\int \frac{d\text{LIPS}_{12}}{d\phi_1 d\cos\theta_2 d\phi_2} d\phi_1 d\cos\theta_2 d\phi_2 = \frac{1}{32\pi^3} \delta(E - E_1 - E_2 - E_3) \frac{d\cos\theta_1 |\mathbf{p}_1|^2 d|\mathbf{p}_1| |\mathbf{p}_2|^2 d|\mathbf{p}_2|}{E_1 E_2 E_3}. \quad (6)$$

And on we go...

Finally, integrate over θ_1 , the polar angle of particle 1 (the angle between particle 1 and particle 2). Express your result as a differential in E_1 and E_2 .

Note that the only “physics” you need for this problem is the relation between the rest mass, energy and magnitude of three momentum of a particle:

$$E_i^2 = m_i^2 + |\mathbf{p}_i|^2. \quad (7)$$

Solution: This last part is the hardest part. We want to use the remaining delta function, but it is a delta function in energy, not in $\cos\theta_1$. But $\cos\theta_1$ comes into the expression for E_3 in this delta function:

$$E_3 = \sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}. \quad (8)$$

Thus, we have a delta function of the form $\delta[f(\cos\theta_1)]$, where

$$f(\cos\theta_1) = E - E_1 - E_2 - \sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}. \quad (9)$$

We need the derivative of this with respect to $\cos\theta_1$:

$$\frac{df(\cos\theta_1)}{d\cos\theta_1} = \frac{|\mathbf{p}_1||\mathbf{p}_2|}{\sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}}. \quad (10)$$

We must take the absolute value of this derivative evaluated at the value of $\cos\theta_1$ where $f = 0$, that is at:

$$\sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1} = E - E_1 - E_2. \quad (11)$$

Let $\cos \theta_{10}$ be the solution to this. Note that we won't have to actually solve for $\cos \theta_{10}$. Using

$$\delta(E - E_1 - E_2 - E_3) = \delta(\cos \theta_1 - \cos \theta_{10}) / \left| \frac{df}{d \cos \theta_1}(\cos \theta_{10}) \right|, \quad (12)$$

we obtain

$$\int \frac{d\text{LIPS}_{12}}{d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2} d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2 = \frac{1}{32\pi^3} \frac{|\mathbf{p}_1| |d|\mathbf{p}_1| |\mathbf{p}_2| |d|\mathbf{p}_2|}{E_1 E_2}. \quad (13)$$

We are asked to express our answer as a differential in E_1 and E_2 . This is accomplished with:

$$\frac{|\mathbf{p}_i| |d|\mathbf{p}_i|}{E_i} = \frac{d|\mathbf{p}_i|^2}{2E_i} \quad (14)$$

$$= \frac{dE_i^2}{2E_i} \quad (15)$$

$$= dE_i. \quad (16)$$

The result is:

$$\int \frac{d\text{LIPS}_{12}}{d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2} d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2 = \frac{1}{32\pi^3} dE_1 dE_2. \quad (17)$$

2. In this note we considered the set of test functions:

$$\tau(x; a) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & |x| < a, \\ 0 & |x| > a \end{cases} \quad (18)$$

where a is a positive real number. Consider making a Taylor series expansion about $x = a$. What happens? In spite of the “niceness” of this function, you should encounter a problem. To understand this, show that, as a function of the complex variable z , $\tau(z; a)$ is not analytic at $z = a$. What is the nature of the singularity at $z = a$ (removable?, simple?, order n ?, essential?, isolated?, not isolated?)?

3. Prove that

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x), \quad (19)$$

where $f(x)$ is an infinitely differentiable function. Convince yourself that it makes “intuitive” sense.

Solution: We'll start with

$$(\delta', t) = -(\delta, t'). \quad (20)$$

If $f(x)$ is infinitely differentiable, and if $t \in \mathcal{T}$, then $ft \in \mathcal{T}$. Hence, we can write:

$$(f\delta', t) = (\delta', ft) \quad (21)$$

$$= -(\delta, f't + ft') \quad (22)$$

$$= -f'(0)t(0) - f(0)t'(0) \quad (23)$$

$$= -(f'(0)\delta, t) + (f(0)\delta', t). \quad (24)$$

Hence, the asserted property is demonstrated.

4. Consider the differential equation

$$u''(x) = a(x), \quad (25)$$

where $x \in (-\infty, \infty)$ and $a(x)$ is some known function of x . By considering solutions to

$$\frac{\partial^2 G(x, y)}{\partial x^2} = \delta(x - y), \quad (26)$$

do the following:

(a) Solve for $u(x)$ subject to the boundary conditions:

$$u(0) = u'(0) = 0. \quad (27)$$

Solution: Once we find G , the solution to the differential equation will be given by:

$$u(x) = \int_{-\infty}^{\infty} G(x, y)a(y)dy. \quad (28)$$

We note that for $x \neq y$, $G(x, y)$ is of the form $\alpha + \beta x$. But the constants may be different depending on whether $x < y$ or $x > y$. We thus write in general:

$$G(x, y) = \begin{cases} (A - \alpha) + (B - \beta)x, & x \leq y, \\ (A + \alpha) + (B + \beta)x, & x \geq y, \end{cases} \quad (29)$$

where the “constants” A, B, α, β may depend on y .

There are certain constraints we can impose before considering the boundary conditions. First, G must be continuous at $x = y$. Therefore

$$A - \alpha + (B - \beta)y = A + \alpha + (B + \beta)y, \quad (30)$$

or $\alpha(y) = -\beta(y)y$.

$$G(x, y) = \begin{cases} (A + \beta y) + (B - \beta)x, & x \leq y, \\ (A - \beta y) + (B + \beta)x, & x \geq y, \end{cases} \quad (31)$$

Second, the derivative of G must have a step discontinuity at $x = y$ in order for the second derivative to give the delta functional:

$$(B + \beta) - (B - \beta) = 1, \quad (32)$$

or $\beta = 1/2$. Thus,

$$G(x, y) = A + Bx + \begin{cases} \frac{y-x}{2}, & x \leq y, \\ \frac{x-y}{2}, & x \geq y. \end{cases} \quad (33)$$

Now we are ready for the boundary conditions. Let's consider $u(0) = 0$ first:

$$G(0, y) = 0 = A + \begin{cases} \frac{y}{2}, & 0 \leq y, \\ -\frac{y}{2}, & 0 \geq y. \end{cases} \quad (34)$$

Thus, $A = -y/2$ for $y \geq 0$, and $A = y/2$ for $y \leq 0$. The boundary condition on the derivative gives:

$$\frac{\partial G}{\partial x}(0, y) = 0 = B + \begin{cases} -\frac{1}{2}, & 0 \leq y, \\ \frac{1}{2}, & 0 \geq y. \end{cases} \quad (35)$$

Thus, $B = 1/2$ for $y \geq 0$, and $B = -1/2$ for $y \leq 0$. Finally, the Green's function may be written compactly as:

$$G(x, y) = [\theta(y) + \theta(x - y) - 1](x - y) \quad (36)$$

The solution to the original problem is thus:

$$u(x) = \int_{-\infty}^{\infty} [\theta(y) + \theta(x - y) - 1](x - y)a(y)dy \quad (37)$$

$$= \begin{cases} \int_x^0 (y - x)a(y)dy & x \leq 0 \\ \int_0^x (x - y)a(y)dy & x \geq 0. \end{cases} \quad (38)$$

(b) Solve for $u(x)$ subject to the boundary conditions:

$$u(0) = \int_0^1 u(x)dx = 0. \quad (39)$$

Solution: [Thanks to Chan Y. Park for a correction] From part (a), we have the Green's function up through the $u(0) = 0$ boundary condition:

$$G(x, y) = \left[\frac{1}{2} - \theta(y)\right]y + Bx + \begin{cases} \frac{y-x}{2}, & x \leq y, \\ \frac{x-y}{2}, & x \geq y, \end{cases} \quad (40)$$

or

$$G(x, y) = \left[\frac{1}{2} - \theta(y)\right]y + Bx + \left[\theta(x - y) - \frac{1}{2}\right](x - y). \quad (41)$$

For the other boundary conditions, we consider the following cases:

$y \leq 0$: In this case,

$$0 = \int_0^1 \left(\frac{y}{2} + Bx + \frac{x-y}{2} \right) dy = \frac{B}{2} + \frac{1}{4}, \quad (42)$$

yielding $B = -1/2$.

$y \geq 1$: In this case, we find $B = 1/2$.

$0 < y < 1$: In this case,

$$0 = \int_0^1 \left(-\frac{y}{2} + Bx \right) dx + \frac{1}{2} \int_0^y (y-x) dx + \frac{1}{2} \int_y^1 (x-y) dx. \quad (43)$$

This gives $B = 2y - y^2 - \frac{1}{2}$.

We can summarize $G(x, y)$ as

$$G(x, y) = (x-y) [\theta(x-y) - \theta(-y)] - (1-y)^2 x \theta(y) \theta(1-y). \quad (44)$$

From this $G(x, y)$, we get

$$u(x) = \int_{-\infty}^{\infty} G(x, y) a(y) dy \quad (45)$$

$$= \int_0^x (x-y) a(y) dy - x \int_0^1 (1-y)^2 a(y) dy. \quad (46)$$

(c) Can $u''(x) = a(x)$ be solved subject to the boundary conditions:

$$u(0) = u(1), \quad \text{and} \quad u'(0) = u'(1)? \quad (47)$$

Solution: We can go back to the Green's function before imposing boundary conditions:

$$G(x, y) = A + Bx + \begin{cases} \frac{y-x}{2}, & x \leq y, \\ \frac{x-y}{2}, & x \geq y. \end{cases} \quad (48)$$

Suppose $0 < y < 1$. The first boundary condition implies

$$A - y/2 = A + B + (y-1)/2, \quad (49)$$

or $B = \frac{1}{2} - y$. The derivative of G is:

$$\frac{\partial G}{\partial x}(x, y) = B + \begin{cases} -\frac{1}{2}, & x \leq y, \\ \frac{1}{2}, & x \geq y. \end{cases} \quad (50)$$

Hence, the second boundary condition yields $B + 1/2 = B - 1/2$. There is no way to satisfy the boundary condition.

We note, however, that there is a solution to the homogenous equation that satisfies these boundary conditions, namely $u_0(x) = c$, where c is any constant. Thus, if $a(x)$ is orthogonal to $u_0(x)$, there will be a solution to the differential equation, but it will not be unique.