

**Physics 129a**  
**Introduction to Distributions**  
**Solutions to Problems**  
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**Revision 081021**

## 1 Exercises

1. In class we tried an example of integrating over a “delta function” involving two-body Lorentz Invariant Phase Space (LIPS). Let us try it for three body LIPS. Thus, suppose we start with an initial energy  $E$  and three-momentum  $\mathbf{p} = \mathbf{0}$ . We write the phase space volume element as:

$$dLIPS = (2\pi)^4 \delta^{(4)}(p - p_1 - p_2 - p_3) \frac{d^3(\mathbf{p}_1)}{(2\pi)^3 2E_1} \frac{d^3(\mathbf{p}_2)}{(2\pi)^3 2E_2} \frac{d^3(\mathbf{p}_3)}{(2\pi)^3 2E_3}, \quad (1)$$

where  $p, p_1, p_2, p_3$  denote four-vectors. Integrate this differential over  $d^3(\mathbf{p}_3)$  to get a differential of the form:

$$dLIPS_{12} = \int \frac{dLIPS}{d^3(\mathbf{p}_3)} d^3(\mathbf{p}_3). \quad (2)$$

**Solution:** This part is easy:

$$dLIPS_{12} = \int \frac{dLIPS}{d^3(\mathbf{p}_3)} d^3(\mathbf{p}_3) \quad (3)$$

$$= \frac{1}{8(2\pi)^5} \delta(E - E_1 - E_2 - E_3) \frac{d^3(\mathbf{p}_1) d^3(\mathbf{p}_2)}{E_1 E_2 E_3}, \quad (4)$$

where now  $E_3 = \sqrt{m_3^2 + (\mathbf{p}_1 + \mathbf{p}_2)^2}$ . On to the next piece...

This is a differential in  $d^3(\mathbf{p}_1) d^3(\mathbf{p}_2)$ . Assume that you may pick a coordinate system orientation arbitrarily (that is, there is no preferred direction specified in the initial state). With this symmetry, integrate over the angular variables of particle two.

**Solution:** Write  $d^3(\mathbf{p}_2) = |\mathbf{p}_2|^2 d|\mathbf{p}_2| d\cos\theta_2 d\phi_2$ . The fact that we can pick our coordinate orientation arbitrarily means that the physics must be independent of the direction of particle two. Thus, we integrate over  $\cos\theta_2$  and  $\phi_2$  to get a factor of  $4\pi$ :

$$\int \frac{dLIPS_{12}}{d\cos\theta_2 d\phi_2} d\cos\theta_2 d\phi_2 = \frac{1}{4(2\pi)^4} \delta(E - E_1 - E_2 - E_3) \frac{d^3(\mathbf{p}_1) |\mathbf{p}_2|^2 d|\mathbf{p}_2|}{E_1 E_2 E_3}, \quad (5)$$

Next part...

You should have remaining a differential quantity in  $d^3(\mathbf{p}_1)d|\mathbf{p}_2|$ , where the angular variables of particle 1 are measured with respect to the particle 2 direction. Assume that there is no special azimuthal direction determined by particle 2, and integrate over the azimuth angle  $\phi_1$ , of particle 1.

**Solution:** Again, we are using the assumption that the initial state picks out no special directions to argue that the physics is independent of the azimuth angle of particle one in the coordinate system defined by particle 2. Write  $d^3(\mathbf{p}_1) = |\mathbf{p}_1|^2 d|\mathbf{p}_1| d\cos\theta_1 d\phi_1$ . Integrate over  $\phi$  to get a factor of  $2\pi$ :

$$\int \frac{d\text{LIPS}_{12}}{d\phi_1 d\cos\theta_2 d\phi_2} d\phi_1 d\cos\theta_2 d\phi_2 = \frac{1}{32\pi^3} \delta(E - E_1 - E_2 - E_3) \frac{d\cos\theta_1 |\mathbf{p}_1|^2 d|\mathbf{p}_1| |\mathbf{p}_2|^2 d|\mathbf{p}_2|}{E_1 E_2 E_3}. \quad (6)$$

And on we go...

Finally, integrate over  $\theta_1$ , the polar angle of particle 1 (the angle between particle 1 and particle 2). Express your result as a differential in  $E_1$  and  $E_2$ .

Note that the only “physics” you need for this problem is the relation between the rest mass, energy and magnitude of three momentum of a particle:

$$E_i^2 = m_i^2 + |\mathbf{p}_i|^2. \quad (7)$$

**Solution:** This last part is the hardest part. We want to use the remaining delta function, but it is a delta function in energy, not in  $\cos\theta_1$ . But  $\cos\theta_1$  comes into the expression for  $E_3$  in this delta function:

$$E_3 = \sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}. \quad (8)$$

Thus, we have a delta function of the form  $\delta[f(\cos\theta_1)]$ , where

$$f(\cos\theta_1) = E - E_1 - E_2 - \sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}. \quad (9)$$

We need the derivative of this with respect to  $\cos\theta_1$ :

$$\frac{df(\cos\theta_1)}{d\cos\theta_1} = \frac{|\mathbf{p}_1||\mathbf{p}_2|}{\sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1}}. \quad (10)$$

We must take the absolute value of this derivative evaluated at the value of  $\cos\theta_1$  where  $f = 0$ , that is at:

$$\sqrt{m_3^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_1} = E - E_1 - E_2. \quad (11)$$

Let  $\cos \theta_{10}$  be the solution to this. Note that we won't have to actually solve for  $\cos \theta_{10}$ . Using

$$\delta(E - E_1 - E_2 - E_3) = \delta(\cos \theta_1 - \cos \theta_{10}) / \left| \frac{df}{d \cos \theta_1}(\cos \theta_{10}) \right|, \quad (12)$$

we obtain

$$\int \frac{d\text{LIPS}_{12}}{d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2} d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2 = \frac{1}{32\pi^3} \frac{|\mathbf{p}_1| |d|\mathbf{p}_1| |\mathbf{p}_2| |d|\mathbf{p}_2|}{E_1 E_2}. \quad (13)$$

We are asked to express our answer as a differential in  $E_1$  and  $E_2$ . This is accomplished with:

$$\frac{|\mathbf{p}_i| |d|\mathbf{p}_i|}{E_i} = \frac{d|\mathbf{p}_i|^2}{2E_i} \quad (14)$$

$$= \frac{dE_i^2}{2E_i} \quad (15)$$

$$= dE_i. \quad (16)$$

The result is:

$$\int \frac{d\text{LIPS}_{12}}{d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2} d \cos \theta_1 d \phi_1 d \cos \theta_2 d \phi_2 = \frac{1}{32\pi^3} dE_1 dE_2. \quad (17)$$

2. In this note we considered the set of test functions:

$$\tau(x; a) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & |x| < a, \\ 0 & |x| > a \end{cases} \quad (18)$$

where  $a$  is a positive real number. Consider making a Taylor series expansion about  $x = a$ . What happens? In spite of the “niceness” of this function, you should encounter a problem. To understand this, show that, as a function of the complex variable  $z$ ,  $\tau(z; a)$  is not analytic at  $z = a$ . What is the nature of the singularity at  $z = a$  (removable?, simple?, order  $n$ ?, essential?, isolated?, not isolated?)?

3. Prove that

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x), \quad (19)$$

where  $f(x)$  is an infinitely differentiable function. Convince yourself that it makes “intuitive” sense.

**Solution:** We'll start with

$$(\delta', t) = -(\delta, t'). \quad (20)$$

If  $f(x)$  is infinitely differentiable, and if  $t \in \mathcal{T}$ , then  $ft \in \mathcal{T}$ . Hence, we can write:

$$(f\delta', t) = (\delta', ft) \tag{21}$$

$$= -(\delta, f't + ft') \tag{22}$$

$$= -f'(0)t(0) - f(0)t'(0) \tag{23}$$

$$= -(f'(0)\delta, t) + (f(0)\delta', t). \tag{24}$$

Hence, the asserted property is demonstrated.