

Chapter 3

Representation Theory

Groups may be very abstract objects and operations in general, and it would be convenient if we could always put them in some standard, equivalent form, and in particular a form that lends itself to easy manipulation. This is the motivation for the following discussion.

Def: Let (G, \circ) and $(H, *)$ be two groups. These groups are called *isomorphic* ($G \cong H$) if:

1. There exists a one-to-one mapping $\phi : G \rightarrow H$ from G onto H such that
2. $\phi(a \circ b) = \phi(a) * \phi(b), \forall a, b \in G$.

Note that (1) implies that G and H have the same order, and (2) implies that the multiplication tables of G and H are “identical”, up to relabeling of elements (as specified by the mapping ϕ). Statements about the structure of group G are equivalent to statements about the structure of group H , and we may choose to study either case for convenience.

A somewhat less rigid correspondence is given by:

Def: Let (G, \circ) and $(H, *)$ be two groups. A mapping ϕ from G into H is called a *homomorphism* if: $\phi(a \circ b) = \phi(a) * \phi(b), \forall a, b \in G$.

In this case, the mapping may be many-to-one. An extreme case occurs with H being a group of order one (the identity element), and all elements of G are mapped into this element of H .

Def: An isomorphism of a group into itself is called an *automorphism*. A homomorphism of a group into itself is called an *endomorphism*.

In physical applications, the most prevalent isomorphisms and homomorphisms of abstract groups are into matrix groups:

Def: A (matrix) *representation* of a group G is a group of matrices (with group multiplication given by matrix multiplication) obtained by a homomorphism of G into the set of $n \times n$ matrices. A matrix representation which is an isomorphism is called a *faithful representation* of the group.

It is perhaps worth demonstrating that if G is represented by non-singular matrices D , then $D(e) = I$ is the identity matrix: We start by noting that $D(e)^2 = D(ee) = D(e)$. Now,

$$\begin{aligned} D(g^{-1}) &= D(g^{-1})I \\ &= D(g^{-1})D(g)D(g)^{-1} \\ &= D(g^{-1}g)D(g)^{-1} \\ &= D(e)D(g)^{-1}. \end{aligned}$$

Multiplying by $D(e)$, we find, $D(e)D(g^{-1}) = D(e)D(g)^{-1}$. Multiply both sides now by $D(e)^{-1}$, to find that $D(g^{-1}) = D(g)^{-1}$. Thus, we see that, in a non-singular representation, the matrix representing the inverse of a group element is just the inverse of the matrix representing the original group element. In particular,

$$\begin{aligned} D(e) &= D(g)D(g^{-1}) \\ &= D(g)D(g)^{-1} \\ &= I. \end{aligned}$$

3.1 Poincaré Group

An important example of a group representation is the representation of the Poincaré group with a set of 5×5 matrices. The Poincaré group is also known as the inhomogeneous Lorentz group, and is denoted by \bar{L} . It is the group consisting of all homogeneous Lorentz transformations (velocity boosts, rotations, and reflections, including time-reversal), plus all translations in spacetime. The abstract group \bar{L} may be represented by the set of all 5×5 matrices of the form:

$$\Lambda(M, z) = \begin{pmatrix} & & & & z_1 \\ & & & & z_2 \\ & M & & & z_3 \\ & & & & z_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

where M is a 4×4 matrix which “preserves the invariant interval”¹ when multiplying 4-vectors, and $z = (z_1, z_2, z_3, z_4)$ is any element of R^4 .

¹The invariant interval (squared) between vectors a and b is $(a - b)^2 = (a_4 - b_4)^2 - (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2$.

Thus, an inhomogeneous Lorentz transformation is a transformation of the form:

$$\{x'\} = \Lambda(M, z)\{x\} = \{Mx + z\}, \quad (3.2)$$

where we use the artifice for any four-vector x :

$$\{x\} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 1 \end{pmatrix}. \quad (3.3)$$

This permits us to express an inhomogeneous transformation as a linear transformation.

Our representation is actually isomorphic to \bar{L} . Note, however, that the $n \times n$ identity matrix (pick any n) is also a representation for \bar{L} , although no longer an isomorphism. In fact, the $n \times n$ unit matrix is a representation for any group, although “trivial”.

There are some important subgroups of the Poincaré group, such as:

1. The group Tr , of all pure translations in spacetime is a proper subgroup of \bar{L} . A representation for Tr is the set of matrices of the form:

$$\Lambda(I, z) = \begin{pmatrix} 1 & 0 & 0 & 0 & z_1 \\ 0 & 1 & 0 & 0 & z_2 \\ 0 & 0 & 1 & 0 & z_3 \\ 0 & 0 & 0 & 1 & z_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.4)$$

2. The group L , of homogeneous Lorentz transformations is another subgroup of \bar{L} , with a representation:

$$\Lambda(M, 0) = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & M & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.5)$$

Note that both $\Lambda(M, 0)$ and M itself provide representations for L .

Intuitively, we might suppose that Tr is in fact an invariant subgroup of \bar{L} . For example, if we first perform a rotation, then do a translation, then “undo” the rotation, we think the overall result should be a translation. See Fig. 3.1. However, our intuition may become strained when we include boosts and reflections, so let us see whether we can make a convincing demonstration.

If Tr is an invariant subgroup of \bar{L} , we must show that:

$$\Lambda^{-1}(M, z')\Lambda(I, z)\Lambda(M, z') \in Tr, \quad \forall \Lambda(M, z') \in \bar{L}. \quad (3.6)$$

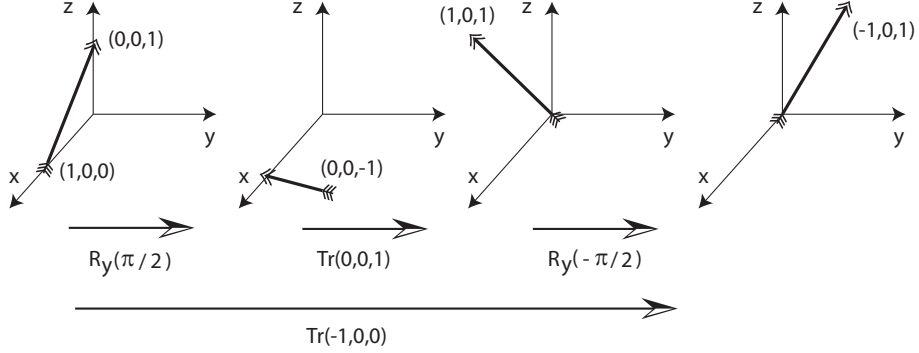


Figure 3.1: Illustration suggesting that $R_y(-\pi/2)Tr(0,0,1)R_y(\pi/2) = Tr(-1,0,0)$.

So far, we have avoided knowing the “multiplication table” for \bar{L} . But now life would be much easier if we knew it. So, what is

$$\Lambda(M', z')\Lambda(M'', z'')? \quad (3.7)$$

We could find the answer by considering the faithful representation:

$$\Lambda(M, z) = \begin{pmatrix} & & z_1 \\ & M & z_2 \\ & & z_3 \\ & & z_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.8)$$

and seeing what ordinary matrix multiplication gives us. The reader is encouraged to try this.

However, it is perhaps more instructive to remember that the elements of \bar{L} are transformations in spacetime, and approach the question by looking at the action of $\Lambda \in \bar{L}$ on an arbitrary 4-vector. Thus, recalling that $\Lambda(M, z)\{x\} = \{Mx + z\}$, we have:

$$\begin{aligned} \Lambda(M', z')\Lambda(M'', z'')\{x\} &= \Lambda(M', z')\{M''x + z''\} \\ &= \{M'(M''x + z'') + z'\} \\ &= \Lambda(M'M'', M'z'' + z')\{x\}. \end{aligned} \quad (3.9)$$

This relation holds for any 4-vector x , hence we have the multiplication table:

$$\Lambda(M', z')\Lambda(M'', z'') = \Lambda(M'M'', M'z'' + z'). \quad (3.10)$$

To see whether Tr is an invariant subgroup of \bar{L} , we also need to know the inverse of $\Lambda \in \bar{L}$: What is $\Lambda^{-1}(M, z)$? Since the identity element is obviously

the transformation where we “do nothing”, $\Lambda(I, 0)$, we must find $\Lambda^{-1} \in \bar{L}$ such that

$$\Lambda^{-1}(M, z)\Lambda(M, z) = \Lambda(I, 0). \quad (3.11)$$

Since $\Lambda^{-1} \in \bar{L}$, we must be able to parameterize it with a matrix M' and a translation z' : $\Lambda^{-1}(M, z) = \Lambda(M', z')$, so that

$$\Lambda(M', z')\Lambda(M, z) = \Lambda(I, 0). \quad (3.12)$$

Use the multiplication table to obtain:

$$\begin{aligned} \Lambda(I, 0) &= \Lambda(M', z')\Lambda(M, z) \\ &= \Lambda(M'M, M'z + z'). \end{aligned} \quad (3.13)$$

Thus,

$$\Lambda^{-1}(M, z) = \Lambda(M^{-1}, -M^{-1}z). \quad (3.14)$$

We are finally ready to prove that Tr is an invariant subgroup of \bar{L} :

$$\begin{aligned} \Lambda^{-1}(M, z')\Lambda(I, z)\Lambda(M, z') &= \Lambda(M^{-1}, -M^{-1}z')\Lambda(M, z + z') \\ &= \Lambda(M^{-1}M, M^{-1}(z + z') - M^{-1}z') \\ &= \Lambda(I, M^{-1}z) \in Tr \quad \forall \Lambda(M, z') \in \bar{L}. \end{aligned} \quad (3.15)$$

Hence, Tr is an invariant subgroup of \bar{L} .

Given any element $\Lambda(M, 0) \in L$, we have an isomorphism of Tr into Tr :

$$\Lambda(I, z) \rightarrow \Lambda(I, M^{-1}z). \quad (3.16)$$

That is, we have an automorphism on Tr .

3.2 Regular Representation

For any finite group $G = \{g_1, g_2, \dots, g_n\}$, with multiplication table $g_i g_j = g_k$, we may construct an isomorphic representation by a set of $n \times n$ matrices. Consider the following expression:

$$g_i g_j = g_m \Delta_{ij}^m. \quad (3.17)$$

A formal sum over $m = 1, 2, \dots, n$ is implied here. In fact, only one term in the sum is non-zero, with

$$\Delta_{ij}^m = \delta_k^m, \quad (3.18)$$

where δ_k^m is the Kronecker delta, as follows from the group multiplication table (index k is a function of i and j). We have the theorem:

Theorem: The *regular representation*, formed by the matrices $(\Delta_i)_j^k = \Delta_{ij}^k$, $i = 1, 2, \dots, n$, forms an isomorphic representation of G .

Proof: To understand a bit better what is going on, note that the matrices consist entirely of zeros and ones, where the matrix representing group element $a \in G$ has ones in locations such that a times an element of G specified by the column index gives an element of G given by the row index. Consider now:

$$ag_k = g_m \Delta_{a_k}^m, \quad (3.19)$$

where we use the group element a also as its index in the set of elements. Notice that

$$(\Delta_a)_k^m = \delta_{a_k}^m, \quad (3.20)$$

where $g_{a_k} \equiv ag_k$ determines index a_k . Suppose $ab = c$, for $a, b, c \in G$. Let us check that this multiplication table is preserved by our representation:

$$\begin{aligned} (\Delta_a)_m^k (\Delta_b)_j^m &= \delta_{a_m}^k \delta_{b_j}^m \\ &= \delta_{a_{b_j}}^k \\ &= \delta_{c_j}^k \\ &= (\Delta_c)_j^k. \end{aligned} \quad (3.21)$$

This follows since

$$\begin{aligned} g_{a_{b_j}} &\equiv ag_{b_j} \\ &= abg_j \\ &= cg_j \\ &= g_{c_j}, \end{aligned} \quad (3.22)$$

that is, $a_{b_j} = c_j$. The remaining aspects of the proof are left to the reader.

We deal in the following with finite-dimensional representations.

3.3 Equivalence of Representations

Def: Two $n \times n$ matrix representations, D and D' , where $D(g)$ and $D'(g)$ are the matrices representing group element g in the two representations, are said to be *equivalent* if there exists a non-singular matrix S such that

$$D'(g) = S^{-1}D(g)S, \quad \forall g \in G. \quad (3.23)$$

It is readily checked that this defines a true equivalence relation; the reader is encouraged to do so. Note that the matrix S need not be a member of either representation. If D and D' are equivalent representations, we write $D \sim D'$.

A transformation of this form may be regarded as simply a change in basis for the vector space upon which our matrices operate. Hence, equivalent representations are identical as far as the intrinsic internal group structure is concerned. Presuming we are really interested in this intrinsic structure, we

would like to be able to concentrate on those statements which are independent of “coordinate” system. That is, we are interested in studying quantities which are invariant with respect to similarity transformations.

In principle, there are n such invariant quantities, corresponding to the n eigenvalues. However, we typically don’t need to study all n . In fact, just one invariant, the trace (sum of the eigenvalues), contains sufficient information for many purposes. Recall

$$\mathrm{Tr} [D(g)] = \sum_{i=1}^n D_{ii}(g). \quad (3.24)$$

This is invariant under similarity transformations:

$$\begin{aligned} \mathrm{Tr} [D'(g)] &= \mathrm{Tr} [S^{-1}D(g)S] \\ &= \mathrm{Tr} [SS^{-1}D(g)] \\ &= \mathrm{Tr} [D(g)]. \end{aligned} \quad (3.25)$$

3.4 Characters

The trace of a representation matrix plays a very important role, so it gets a special name:

Def: The trace of $D(g)$ is called the *character* of g in the representation D .

The character is usually denoted with the Greek letter chi:

$$\chi(g) = \mathrm{Tr} [D(g)]. \quad (3.26)$$

We have seen that equivalent representations have the same set of characters. We have the further fact:

Theorem: Given a representation D , any two group elements belonging to the same class have the same character.

The proof of this is straightforward: Suppose that g_1 and g_2 belong to the same class in G . Then there exists an element $h \in G$ such that

$$h^{-1}g_1h = g_2. \quad (3.27)$$

Thus, in representation D , we must have:

$$D(h^{-1})D(g_1)D(h) = D(g_2). \quad (3.28)$$

Note also that $D(h)D(h^{-1}) = D(e)$ (we are not assuming that $D(h)^{-1}$ exists here, as our representation could be singular). Thence,

$$\begin{aligned} \chi(g_2) &= \mathrm{Tr} [D(g_2)] \\ &= \mathrm{Tr} [D(h^{-1})D(g_1)D(h)] \end{aligned}$$

$$\begin{aligned}
&= \operatorname{Tr} [D(h)D(h^{-1})D(g_1)] \\
&= \operatorname{Tr} [D(e)D(g_1)] \\
&= \operatorname{Tr} [D(eg_1)] \\
&= \operatorname{Tr} [D(g_1)] \\
&= \chi(g_1).
\end{aligned} \tag{3.29}$$

Thus, given a representation D , we can completely specify the character structure by evaluating the character for one member of each class – the character is a “class function”.

We are often interested in more than one representation for a given group G . In this case, we can add labels to the representations to distinguish them, for example, $D^{(a)}, D^{(b)}, \dots$. We similarly label the characters, e.g., $\chi^{(a)}, \chi^{(b)}, \dots$. If we have a class, say C_i , in representation (a) , we may refer to the “character of the class” as $\chi^{(a)}(C_i)$.

3.5 Unitary Representations

Unitary matrices are especially nice. They preserve the lengths of vectors when operating on a complex vector space. The inverse is easy to compute: If U is a unitary matrix, then $U^{-1} = U^\dagger$, where the \dagger means to take the transpose of the complex conjugate matrix. Thus, it is quite nice to learn that:

Theorem: If G is a finite group, then every non-singular representation (that is, representation by non-singular matrices) is equivalent to a unitary representation.

Thus, at least for finite groups, it is sufficient to consider representations by unitary matrices. The proof of this theorem is instructive:

Proof: We suppose we are given a (non-singular, but possibly non-unitary) $n \times n$ representation, D , of G . We may regard an element of the representation as a linear operator on an n -dimensional vector space. Define a scalar product on the vector space by

$$(x, y) \equiv \sum_{i=1}^n x_i^* y_i, \tag{3.30}$$

where x and y are any pair of vectors.

Suppose that we have a matrix U with the property that:

$$(Ux, Uy) = (x, y), \quad \forall x, y. \tag{3.31}$$

That is, U “preserves the scalar product”. Let us see what this condition requires for U :

$$(Ux, Uy) = \sum_{i=1}^n (Ux)_i^* (Uy)_i$$

$$\begin{aligned}
&= \sum_{i=1}^n \left[\sum_{j=1}^n U_{ij}^* x_j^* \right] \left[\sum_{k=1}^n U_{ik} y_k \right] \\
&= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (U^\dagger)_{ji} U_{ik} x_j^* y_k \\
&= \sum_{j=1}^n \sum_{k=1}^n (U^\dagger U)_{jk} x_j^* y_k \\
&= \sum_{j=1}^n x_j^* y_j \quad (\text{by assumption}). \tag{3.32}
\end{aligned}$$

But x and y are arbitrary vectors, hence we must have

$$(U^\dagger U)_{jk} = \delta_{jk}, \tag{3.33}$$

or $U^\dagger = U^{-1}$, that is U must be a unitary matrix.

We wish to show that our given representation, D , is equivalent to some unitary representation, say D' . For this to be true there must exist a transformation T such that

$$D' = T^{-1}DT, \tag{3.34}$$

where we mean that this transformation is applied to every element of the representation. If we can find a transformation T such that

$$(D'(a)x, D'(a)y) = (x, y), \quad \forall x, y \text{ and } \forall a \in G, \tag{3.35}$$

then by the above discussion we will have found a unitary representation.

We construct a suitable transformation by the following technique, which introduces an approach that will be useful elsewhere as well. Let g be the order of G . Define an ‘‘average’’ over the elements of the group:

$$\{x, y\} \equiv \frac{1}{g} \sum_{a \in G} (D(a)x, D(a)y). \tag{3.36}$$

In a sense $\{x, y\}$ is the average scalar product over all group elements, with respect to representation D , acting on the vectors x and y . Notice that $\{x, y\}$ itself defines a scalar product, since

1. $\{x, x\} \geq 0$ and $\{x, x\} = 0$ if and only if $x = 0$;
2. $\{x, y\} = \{y, x\}^*$;
3. $\{x, cy\} = c\{x, y\}$;
4. $\{x_1 + x_2, y\} = \{x_1, y\} + \{x_2, y\}$.

We remark that it is the first property that requires D to be a non-singular representation.

Now let b be any element of G , and consider:

$$\begin{aligned}
 \{D(b)x, D(b)y\} &= \frac{1}{g} \sum_{a \in G} (D(a)D(b)x, D(a)D(b)y) \\
 &= \frac{1}{g} \sum_{a \in G} (D(ab)x, D(ab)y) \\
 &= \frac{1}{g} \sum_{a \in G} (D(a)x, D(a)y) \\
 &= \{x, y\}.
 \end{aligned} \tag{3.37}$$

The third step is valid because the multiplication table is a Latin square: Summing products ab over all $a \in G$ is the same as summing ab over all $ab \in G$; the only difference is the ordering in the sum. This “invariance” of the sum is a property that will often come in handy.

Thus, we have shown that $D(b)$ is a unitary operator (that is, it preserves the scalar product) with respect to the $\{, \}$ scalar product. It is not necessarily a unitary operator with respect to the $(,)$ scalar product, that is D is not necessarily a unitary matrix representation. Somehow, we would like to find a transformation which takes this desired unitary property under the $\{, \}$ scalar product back into the $(,)$ scalar product. In other words, we wish to transform from a basis suitable for $(,)$ to one suitable for $\{, \}$.

Consider a set of n orthonormal vectors with respect to $(,)$:

$$(u_i, u_j) = \delta_{ij}, \tag{3.38}$$

and a set orthonormal with respect to $\{, \}$:

$$\{v_i, v_j\} = \delta_{ij}. \tag{3.39}$$

Let T be the transformation operator which takes u 's to v 's:

$$v_i = Tu_i. \tag{3.40}$$

An arbitrary vector x may be expanded in the u basis as:

$$x = \sum_{i=1}^n x_i u_i, \tag{3.41}$$

where the x_i are the components in the u basis. Consider the transformed vector Tx :

$$Tx = T \sum_{i=1}^n x_i u_i = \sum_{i=1}^n x_i T u_i = \sum_{i=1}^n x_i v_i. \tag{3.42}$$

Thus, the components of the transformed vector in the new basis v are the same as the components of the un-transformed vector in the old basis u . We have

$$\begin{aligned}\{Tx, Ty\} &= \sum_{i=1}^n \sum_{j=1}^n x_i^* y_j \{v_i, v_j\} \\ &= \sum_{i=1}^n x_i^* y_i \\ &= (x, y).\end{aligned}\tag{3.43}$$

Now consider the representation

$$D' \equiv T^{-1}DT,\tag{3.44}$$

which is equivalent to D . Evaluate the scalar product:

$$\begin{aligned}(D'(a)x, D'(a)y) &= (T^{-1}D(a)Tx, T^{-1}D(a)Ty) \\ &= \{D(a)Tx, D(a)Ty\} \quad (\text{since } \{x, y\} = (T^{-1}x, T^{-1}y)) \\ &= \{Tx, Ty\} \quad (\text{since } D(a) \text{ is a unitary operator wrt } \{, \}) \\ &= (x, y).\end{aligned}\tag{3.45}$$

Hence, $D'(a)$, for any $a \in G$, is a unitary operator with respect to the $(,)$ scalar product. Therefore, $D'(a)$ is a unitary matrix. This completes the proof.

3.6 Reducible and Irreducible Representations

Def: Given any two representations, $D^{(1)}$ and $D^{(2)}$, of a group G , we may construct a new representation simply by forming the matrix direct sum:

$$D(g) = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix} \equiv D^{(1)}(g) \oplus D^{(2)}(g).\tag{3.46}$$

A representation which is equivalent to a representation of this form is called *fully reducible*. A representation which is equivalent to a representation of the form:

$$\begin{pmatrix} D^{(1)}(g) & A(g) \\ 0 & D^{(2)}(g) \end{pmatrix}\tag{3.47}$$

is called *reducible*. A representation which is not reducible is called *irreducible*.

Note that the definition of reducibility is equivalent to the statement that there exists a proper invariant subspace, V_1 , in the Euclidean space operated on by the representation. The further restriction of full reducibility is equivalent to the statement that the orthogonal complement of V_1 is also an invariant subspace.

Theorem: If a reducible representation, $D(G)$, is equivalent to a unitary representation, then $D(G)$ is fully reducible.

Proof: Let $U(G)$ be a unitary representation which is equivalent to $D(G)$, and let V be the Euclidean space operated on by U . By assumption, there exists a proper invariant subspace $V_1 \subset V$ under the actions of $U(G)$. Define an orthonormal basis for V , consisting of the vectors $\{\mathbf{e}_i, i = 1 \dots n\}$, such that the first n_1 basis vectors span V_1 . Let V_2 be the orthogonal complement of V_1 , spanned by basis vectors $\{\mathbf{e}_i, i = n_1+1 \dots n\}$. We demonstrate that V_2 is also an invariant subspace under $U(G)$. Since $U(G)$ is unitary, we have, for any $g \in G$:

$$(U(g)\mathbf{e}_i, U(g)\mathbf{e}_j) = (\mathbf{e}_i, \mathbf{e}_j) \quad (3.48)$$

Suppose $\mathbf{e}_j \in V_1$ and $\mathbf{e}_i \in V_2$. Then $U(g)\mathbf{e}_j \in V_1$, since V_1 is invariant. Further, since $(\mathbf{e}_i, \mathbf{e}_j) = 0$, $U(g)\mathbf{e}_i$ is orthogonal to any vector in V_1 , since we could pick any $\mathbf{e}_j \in V_1$, and the set of all vectors $\{U(g)\mathbf{e}_j | \mathbf{e}_j \in V_1\}$ spans V_1 . Thus, $U(g)\mathbf{e}_i$ is in V_2 . Therefore, V_2 is also an invariant subspace under $U(G)$. QED

Since we will be dealing here with representations which are equivalent to unitary representations, we may assume that our representations are either fully reducible or irreducible. In our study of group structure, two equivalent irreducible representations are not counted as distinct.

The irreducible representations (or “irreps”, for short) are important because an arbitrary representation can be expressed as a *direct sum* of irreps. For illustration,

$$D(g) = \begin{pmatrix} D^{(1)}(g) & & 0 & \\ & D^{(2)}(g) & & \\ & 0 & D^{(3)}(g) & \\ & & & D^{(3)}(g) \end{pmatrix} = D^{(1)}(g) \oplus D^{(2)}(g) \oplus 2D^{(3)}(g). \quad (3.49)$$

Note that the reduction of a representation to irreps may include some irreps multiple times.

There are some important properties of irreps, under the name of “Schur’s lemmas”:

Theorem: If D and D' are irreps of G , and if matrix A satisfies

$$D(g)A = AD'(g), \quad \forall g \in G, \quad (3.50)$$

then either $D \sim D'$ or $A = 0$.

Proof: Note that A may not be a square matrix, as the dimensions of representations D and D' could be different. We may consider D and D' to be sets of operators on vector spaces V and V' , respectively. The range of A is

$$R_A = \{x \in V : x = Ax', \text{ where } x' \in V'\}. \quad (3.51)$$

R_A is an invariant subspace of V , since

$$\begin{aligned} D(g)x &= D(g)Ax', \quad \text{for any } x \in R_A \\ &= AD'(g)x', \quad (\text{by assumption}) \\ &\in R_A, \quad \text{since } D'(g)x' \in V'. \end{aligned} \quad (3.52)$$

But since D is an irrep, this means that either $R_A = V$ or $R_A = \{0\}$ (that is, $A = 0$).

Now consider

$$N' \equiv \{x' \in V' : Ax' = 0\}. \quad (3.53)$$

This is referred to as the *null space* of A in V' . It is an invariant subspace of D' in V' since, if $x' \in N'$, then

$$AD'(g)x' = D(g)Ax' = D(g)0 = 0. \quad (3.54)$$

But D' is irreducible, therefore either $N' = V'$ (hence $A = 0$) or $N' = \{0\}$. If $N' = \{0\}$, then the equation $Ax' = Ay'$ implies $x' = y'$, and the mapping A is one-to-one and onto.

We have so far shown that either A provides an isomorphism between V and V' or $A = 0$. If an isomorphism, then A is invertible, and

$$D(g) = AD'(g)A^{-1}, \quad \forall g \in G. \quad (3.55)$$

That is, D and D' are equivalent representations in this case. We remark that two irreps can be equivalent only if they have the same dimension.

Theorem: D is an irrep if and only if, given matrix A such that

$$AD(g) = D(g)A, \quad \forall g \in G, \quad (3.56)$$

then A is a constant times the identity matrix.

Proof: Consider the eigenvalue equation

$$Ax = \lambda x, \quad (3.57)$$

where $x \in V$. If x is an eigenvector with eigenvalue λ , then

$$AD(g)x = D(g)Ax = \lambda D(g)x \quad \forall g \in G. \quad (3.58)$$

That is, $D(g)x$ is also an eigenvector of A belonging to eigenvalue λ . The subspace of eigenvectors belonging to λ is invariant with respect to D . Hence, there are three possibilities: either D is reducible, or this subspace is V , or the subspace consists only of $x = 0$. If the subspace is V , then A has only one eigenvalue, and $A = \lambda I$. If the subspace is $x = 0$, then $A = 0$.

Note that the second theorem provides a test for irreducibility: Given a representation D , we look for a matrix A such that $AD(g) = D(g)A$ for all $g \in G$, and see whether it must be true that $A = \lambda I$. For example, consider an abelian group G . Certainly any one-dimensional representation is irreducible, since any number A is a constant times 1. Suppose we have a representation of dimension larger than one. Since G is abelian, we must have

$$D(a)D(b) = D(b)D(a), \quad \forall a, b \in G. \quad (3.59)$$

Thus, pick $A = D(a)$ for some element a of G . Then $AD(b) = D(b)A$ for all $b \in G$. But if $A = \lambda I$ for every $a \in G$, then D is reducible. Suppose there exists $a \in G$ such that $A = D(a) \neq \text{constant} \times I$. Then by the lemma, D must be reducible. We have just shown that all irreps of an abelian group must be one-dimensional.

To pursue this example further with a concrete case, consider the abelian group $Z_5 = \{0, 1, 2, 3, 4\}$ (with group multiplication given by addition modulo five). The inequivalent irreducible matrix representations are shown in Table 3.1; note that they are all one-dimensional.

Table 3.1: The irreducible representations of Z_5 .

$g \setminus \text{irrep}$	$D^{(1)}$	$D^{(2)}$	$D^{(3)}$	$D^{(4)}$	$D^{(5)}$
0	1	1	1	1	1
1	1	$e^{2\pi i/5}$	$e^{4\pi i/5}$	$e^{6\pi i/5}$	$e^{8\pi i/5}$
2	1	$e^{4\pi i/5}$	$e^{8\pi i/5}$	$e^{2\pi i/5}$	$e^{6\pi i/5}$
3	1	$e^{6\pi i/5}$	$e^{2\pi i/5}$	$e^{8\pi i/5}$	$e^{4\pi i/5}$
4	1	$e^{8\pi i/5}$	$e^{6\pi i/5}$	$e^{4\pi i/5}$	$e^{2\pi i/5}$

3.7 Orthogonality Theorems

The Schur's lemmas are also useful in proving the very important "orthogonality relations". These theorems are important tools in determining the essential structure of the irreps for a group.

The first theorem may appropriately be referred to as the "general orthogonality relation".

Theorem: If $D^{(i)}$ and $D^{(j)}$ are irreps, where $i \neq j$ label inequivalent irreps, then

$$\sum_{g \in G} D^{(i)}(g)_{\mu\nu} D^{(j)}(g^{-1})_{\alpha\beta} = \frac{h}{\ell_i} \delta_{ij} \delta_{\alpha\nu} \delta_{\beta\mu}, \quad (3.60)$$

where h is the order of the group, and ℓ_i is the dimension of representation $D^{(i)}$.

Proof: Let A be any $\ell_i \times \ell_j$ matrix (ℓ_i rows and ℓ_j columns). Define

$$M_A \equiv \sum_g D^{(i)}(g)AD^{(j)}(g^{-1}). \quad (3.61)$$

Note the use once again of the technique of summing over the group. Now consider

$$\begin{aligned} D^{(i)}(b)M_A &= \sum_g D^{(i)}(b)D^{(i)}(g)AD^{(j)}(g^{-1}) \\ &= \sum_g D^{(i)}(b)D^{(i)}(g)AD^{(j)}(g^{-1})D^{(j)}(b^{-1})D^{(j)}(b) \\ &= \left[\sum_g D^{(i)}(bg)AD^{(j)}((bg)^{-1}) \right] D^{(j)}(b) \\ &= M_A D^{(j)}(b), \quad (\text{by the rearrangement lemma}). \end{aligned} \quad (3.62)$$

By Schur's lemma, either $D^{(i)} \sim D^{(j)}$ (that is, $i = j$) or $M_A = 0$.

If $i \neq j$, then $M_A = 0$. Picking A such that $A_{\nu\alpha} = 1$ and all other elements are zero, we obtain:

$$\sum_g D^{(i)}(g)_{\mu\nu}D^{(j)}(g^{-1})_{\alpha\beta} = 0, \quad \forall \mu, \nu, \alpha, \beta. \quad (3.63)$$

If $i = j$, then we may simplify the notation, letting $D^{(i)} = D^{(j)} = D$. We have:

$$D(b)M_A = M_A D(b), \quad \forall b \in G. \quad (3.64)$$

By the other Schur's lemma, this means that M_A is a multiple of the identity:

$$\sum_g D(g)AD(g^{-1}) = \lambda_A I, \quad (3.65)$$

where the value of the multiple depends on A . Pick matrix A so that $A_{\nu\alpha} = 1$, with all other elements zero. Then

$$\sum_g D(g)_{\mu\nu}D(g^{-1})_{\alpha\beta} = \delta_{\mu\beta}\lambda_{\nu\alpha}, \quad (3.66)$$

where the Kronecker delta gives the components of the identity matrix, and $\lambda_{\nu\alpha}$ is the constant multiplying the identity.

To determine $\lambda_{\nu\alpha}$, set $\mu = \beta$ and sum over μ :

$$\begin{aligned} \sum_{\mu=1}^{\ell_i} \lambda_{\nu\alpha} &= \sum_g \sum_{\mu} D(g)_{\mu\nu}D(g^{-1})_{\alpha\mu} \\ \ell_i \lambda_{\nu\alpha} &= \sum_g [D(g^{-1})D(g)]_{\alpha\nu} \end{aligned}$$

$$\begin{aligned}
&= \sum_g D(g^{-1})_{\alpha\nu} \\
&= h\delta_{\alpha\nu},
\end{aligned} \tag{3.67}$$

where we have used the fact that any irrep that is equivalent to a unitary representation is non-singular, and hence $D(e) = I$. This completes the proof.

Our theorem holds whether the irreps are unitary or not. For a unitary representation we can restate the general orthogonality relation in a more convenient form. For a unitary irrep, we have

$$D^{(j)}(g^{-1}) = D^{(j)}(g)^{-1} = D^{(j)}(g)^\dagger. \tag{3.68}$$

For unitary irreps we can thus rewrite the general orthogonality relation as

$$\sum_{g \in G} D^{(i)}(g)_{\mu\nu} D^{(j)}(g)_{\beta\alpha}^* = \frac{h}{\ell_i} \delta_{ij} \delta_{\alpha\nu} \delta_{\beta\mu}, \tag{3.69}$$

In obtaining some consequences of this theorem, it is useful to regard the group G as generating an h -dimensional vector space, and to interpret $D^{(i)}(g)_{\mu\nu}$ as the “ g th” component of a vector in this space. The labels i, μ, ν identify a particular vector. The theorem tells us that all such distinct vectors in the space are orthogonal. Let us count how many distinct vectors there are: For a given representation, there are ℓ_i^2 pairs μ, ν , so the number of distinct vectors is

$$\sum_{i=1}^{n_r} \ell_i^2,$$

where n_r is the number of (inequivalent) irreducible representations. Since it is an h -dimensional space, we cannot have more than h linearly independent vectors, hence,

$$\sum_{i=1}^{n_r} \ell_i^2 \leq h. \tag{3.70}$$

In fact, we will soon see that equality holds.²

This equality is a very useful fact to know in approaching the problem of finding irreducible representations. For example, suppose we have a group of order 6. In this case the possible dimensions of the irreducible representations are: (i) $\{\ell_i\} = \{1, 1, 1, 1, 1, 1\}$, corresponding to an abelian group, isomorphic to Z_6 ; and (ii) $\{\ell_i\} = \{1, 1, 2\}$, which can be shown to correspond to the lowest-order non-abelian group. There are no other possibilities for a group of order 6.

The general orthogonality relation yields some subsidiary orthogonality relations for characters, which are very important in evaluating and using character tables.

²We will shortly prove equality by considering the identity element in the regular representation, and showing that its reduction into irreps is such that each irrep occurs in the regular representation a number of times that is equal to the dimension of the irrep.

Theorem: (*First Orthogonality Relation*) Given a group G of order h , with n_c classes and N_k elements in class k , then for unitary irreps $D^{(i)}$ and $D^{(j)}$:

$$\sum_{k=1}^{n_c} \chi^{(i)}(C_k)^* \chi^{(j)}(C_k) N_k = h \delta_{ij}. \quad (3.71)$$

Proof: Start with the general orthogonality relation for irreducible (unitary) representations $D^{(i)}$ and $D^{(j)}$:

$$\sum_{g \in G} D^{(i)}(g)_{\mu\nu}^* D^{(j)}(g)_{\alpha\beta} = \frac{h}{\ell_i} \delta_{ij} \delta_{\mu\alpha} \delta_{\nu\beta}.$$

We are interested in characters, so let $\mu = \nu, \alpha = \beta$, and sum over μ and α :

$$\begin{aligned} \sum_{g \in G} \chi^{(i)}(g)^* \chi^{(j)}(g) &= \frac{h}{\ell_i} \delta_{ij} \sum_{\mu=1}^{\ell_i} \sum_{\alpha=1}^{\ell_i} \delta_{\mu\alpha} \delta_{\mu\alpha} \\ &= h \delta_{ij}. \end{aligned}$$

We complete the proof by replacing the summation $\sum_{g \in G}$ with $\sum_{k=1}^{n_c} N_k$. QED

As with the general orthogonality relation, we may make a geometrical interpretation: Distinct vectors in a space of dimension equal to the number of classes (n_c) are orthogonal (with respect to “weight” N_k). But now the distinct vectors are labelled only by the index (i), and so there are only n_r (the number of irreducible representations) of them. Since the number of distinct vectors cannot exceed the dimension of the space, we have $n_r \leq n_c$.

Our first orthogonality relation tells us that, for irreducible representations $D^{(i)}$, the vectors,

$$\chi^{(i)} = \left(\chi^{(i)}(C_1), \chi^{(i)}(C_2), \dots, \chi^{(i)}(C_{n_c}) \right), \quad i = 1, \dots, n_r, \quad (3.72)$$

form a set of n_r orthogonal vectors (with respect to weight N_k) and hence span an n_r -dimensional subspace of an n_c -dimensional space. Note that the weight N_k poses no essential difficulty, since we could always absorb it into the definition of the vectors if we choose: $\chi^{(i)}(C_k) \rightarrow \chi^{(i)}(C_k) \sqrt{N_k}$.

An arbitrary vector in our subspace may be expanded according to:

$$\chi = \sum_{i=1}^{n_r} a_i \chi^{(i)}. \quad (3.73)$$

In fact, the character of an arbitrary representation may be so expanded, since

$$D = \oplus_{i=1}^{n_r} a_i D^{(i)}, \quad (3.74)$$

which, upon taking the trace of both sides, yields Eq. 3.73. By components, this is:

$$\chi(C_k) = \sum_{i=1}^{n_r} a_i \chi^{(i)}(C_k). \quad (3.75)$$

We can define the inner product between any two vectors by:

$$\lambda \cdot \chi = \sum_{k=1}^{n_c} \lambda(C_k) \chi(C_k)^* N_k. \quad (3.76)$$

To find the expansion coefficients, a_i , take:

$$\begin{aligned} \chi \cdot \chi^{(j)} &= \sum_{i=1}^{n_r} a_i \chi^{(i)} \cdot \chi^{(j)} \\ &= \sum_{i=1}^{n_r} a_i \sum_{k=1}^{n_c} \chi^{(i)}(C_k) \cdot \chi^{(j)}(C_k)^* N_k \\ &= \sum_{i=1}^{n_r} a_i h \delta_{ij} \quad \text{first orthogonality relation} \\ &= a_j h. \end{aligned} \quad (3.77)$$

Thus, we have:

$$a_i = \frac{1}{h} \sum_{k=1}^{n_c} N_k \chi(C_k) \chi^{(i)}(C_k)^*. \quad (3.78)$$

This permits us to prove the following:

Theorem: In the regular representation D of a group of order h , each irreducible representation appears exactly ℓ_i times. Furthermore,

$$\sum_{i=1}^{n_r} \ell_i^2 = h. \quad (3.79)$$

Proof: Recall that the regular representation consists of the matrices (with k, j labelling components):

$$\{\Delta_{ij}^k, i = 1, \dots, h\},$$

where, if $g_i g_j = g_k$ then

$$\Delta_{ij}^m = \begin{cases} 1 & m = k \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\chi(g) = \begin{cases} h & g = e \\ 0 & \text{otherwise,} \end{cases} \quad (3.80)$$

since the regular representation of the identity is the $h \times h$ identity matrix, and if $g \neq e$, then all diagonal elements of the regular representation are zero, by the fact that if $gf = f$, then $g = e$.

Now consider the expansion of the regular representation in terms of irreducible representations:

$$\chi = \sum_{i=1}^{n_r} a_i \chi^{(i)}.$$

Using

$$a_i = \frac{1}{h} \sum_{k=1}^{n_c} \chi(C_k) \chi^{(i)}(C_k)^* N_k,$$

we find that $a_i = \ell_i$, because the irreducible representation of the identity is the $\ell_i \times \ell_i$ unit matrix. Hence, each irreducible representation occurs exactly ℓ_i times in the regular representation.

Finally, since $h = \chi(e) = \sum_{i=1}^{n_r} \ell_i \chi^{(i)}(e)$, we find $\sum_{i=1}^{n_r} \ell_i^2 = h$, which completes the proof. QED

We are now ready to obtain the “second orthogonality relation”:

Theorem: (*Second Orthogonality Relation*) Given a group G of order h , with n_r irreducible unitary representations, we have:

$$\sum_{i=1}^{n_r} \chi^{(i)}(C_k)^* \chi^{(i)}(C_m) = \frac{h}{N_k} \delta_{km}, \quad (3.81)$$

and $n_r = n_c$.

Proof: From the general orthogonality relation, we have $\sum_{i=1}^{n_r} \ell_i^2 = h$ orthonormal (up to a factor of h/ℓ_i) vectors labelled by i, μ, ν , with h components labelled by g . Since there are h vectors, and the space is h -dimensional, this is a complete orthonormal set, which we can express by:

$$\sum_{i=1}^{n_r} \sum_{\mu=1}^{\ell_i} \sum_{\nu=1}^{\ell_i} \frac{\ell_i}{h} D^{(i)}(g)_{\mu\nu}^* D^{(i)}(g')_{\mu\nu} = \delta_{gg'}. \quad (3.82)$$

That is, the sum of the projection operators onto each of the orthogonal directions is the identity matrix. Sum this expression over $g \in C_k$, and $g' \in C_m$ to obtain:

$$\sum_{i=1}^{n_r} \sum_{\mu=1}^{\ell_i} \sum_{\nu=1}^{\ell_i} \frac{\ell_i}{h} \sum_{g \in C_k} D^{(i)}(g)_{\mu\nu}^* \sum_{g' \in C_m} D^{(i)}(g')_{\mu\nu} = \delta_{km} N_k. \quad (3.83)$$

To sum the left-hand side we use:

$$\sum_{g \in C_k} D^{(i)}(g) = \frac{N_k}{\ell_i} \chi^{(i)}(C_k) I. \quad (3.84)$$

Let us quickly demonstrate this. Let

$$S \equiv \sum_{g \in C_k} D^{(i)}(g) \quad (3.85)$$

Consider, for any $a \in G$:

$$\begin{aligned}
D^{(i)}(a^{-1})SD^{(i)}(a) &= \sum_{g \in C_k} D^{(i)}(a^{-1})D^{(i)}(g)D^{(i)}(a) \\
&= \sum_{g \in C_k} D^{(i)}(a^{-1}ga) \\
&= S,
\end{aligned} \tag{3.86}$$

where the final step follows because $a^{-1}ga \in C_k$ and the sum is the same for any $a \in g$. Thus, $SD^{(i)}(a) = D^{(i)}(a)S$ for all $a \in G$, and by Schur's lemma S must therefore be a multiple of the identity. Deriving the constant is left to the reader. We thus obtain:

$$\begin{aligned}
\sum_{i=1}^{n_r} \sum_{\mu=1}^{\ell_i} \sum_{\nu=1}^{\ell_i} \frac{\ell_i}{h} \sum_{g \in C_k} D^{(i)}(g)_{\mu\nu}^* \sum_{g' \in C_m} D^{(i)}(g')_{\mu\nu} &= \sum_{i=1}^{n_r} \sum_{\mu=1}^{\ell_i} \sum_{\nu=1}^{\ell_i} \frac{\ell_i}{h} \frac{N_k N_m}{\ell_i^2} \chi^{(i)}(C_k)^* \chi^{(i)}(C_m) \delta_{\mu\nu} \\
&= \sum_{i=1}^{n_r} \sum_{\mu=1}^{\ell_i} \frac{N_k N_m}{h \ell_i} \chi^{(i)}(C_k)^* \chi^{(i)}(C_m) \\
&= \sum_{i=1}^{n_r} \frac{N_k N_m}{h} \chi^{(i)}(C_k)^* \chi^{(i)}(C_m).
\end{aligned}$$

Substituting into Eq. 3.83, this gives the desired orthogonality relation, Eq. 3.81.

Now, we can interpret Eq. 3.81 as stating that vectors in a n_c -dimensional subspace of an n_r -dimensional space are orthogonal. Hence, $n_r \geq n_c$. But we already have $n_c \geq n_r$, from our discussion following the first orthogonality relation, hence $n_r = n_c$. QED

These theorems are of great help in reducing the effort required to construct and check character tables, which we discuss next.

3.8 Character Tables

If a group G has classes C_1, C_2, \dots, C_{n_c} , then it must have n_c irreducible representations $D^{(1)}, D^{(2)}, \dots, D^{(n_c)}$, with characters $\chi^{(1)}(C_k), \chi^{(2)}(C_k), \dots, \chi^{(n_c)}(C_k)$, $k = 1, \dots, n_c$. We can summarize this in a *character table*, see Table 3.2.

There are several useful things to note concerning a character table:

1. It is a square table, with $n_c = n_r$ rows and $n_c = n_r$ columns.
2. The rows must be orthogonal (remembering to take the complex conjugate of one of them), from the second orthogonality relation.
3. The columns must be orthogonal, with N_k as weighting factors, according to the first orthogonality relation (again remembering complex conjugates).

Table 3.2: Skeleton of a character table.

N_k	$\ell_i \rightarrow$ class \downarrow ; irrep \rightarrow	$\ell_i = 1$ $\chi^{(1)}$	ℓ_2 $\chi^{(2)}$	\cdots \cdots	ℓ_{n_c} $\chi^{(n_c)}$
$N_1 = 1$	$C_1 = \{e\}$	$\ell_1 = 1$	ℓ_2	\cdots	ℓ_{n_c}
N_2	C_2	1	\cdots	\cdots	\cdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
N_{n_c}	C_{n_c}	1	\cdots	\cdots	\cdots

4. By convention, we let C_1 be the class consisting of the identity element. In every irrep the matrix for the identity is the identity matrix. Therefore:

$$\chi^{(k)}(C_1) = \ell_k, \quad (3.87)$$

where ℓ_k is the dimension of irrep k .

5. As demonstrated earlier, we must have

$$\sum_{i=1}^{n_c} \ell_i^2 = h. \quad (3.88)$$

6. The simplest representation of any group is to represent every element by the number one (the “unit” or “identity” representation). This is an irrep, which we by convention here denote $D^{(1)}$. Then the first column of the character table is a string of ones.

Various other facts may be derived and used, but this set is already quite powerful in reducing the amount of work required to construct the character table for a group.

3.9 Decomposition of Reducible Representations

Suppose that we have a representation of a group, which may be reducible. If we have found the character table we may decompose our representation into a direct sum of irreps:

We start by writing the decomposition as:

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus \cdots \oplus a_{n_r} D^{(n_r)}, \quad (3.89)$$

where n_r is the number of irreps, and the a_i are non-negative integers to be determined. Noting that characters are just traces, we have that the character for class C_k must be:

$$\begin{aligned} \chi(C_k) &= \text{Tr}[D(g \in C_k)] \\ &= \sum_{i=1}^{n_r} a_i \chi^{(i)}(C_k). \end{aligned} \quad (3.90)$$

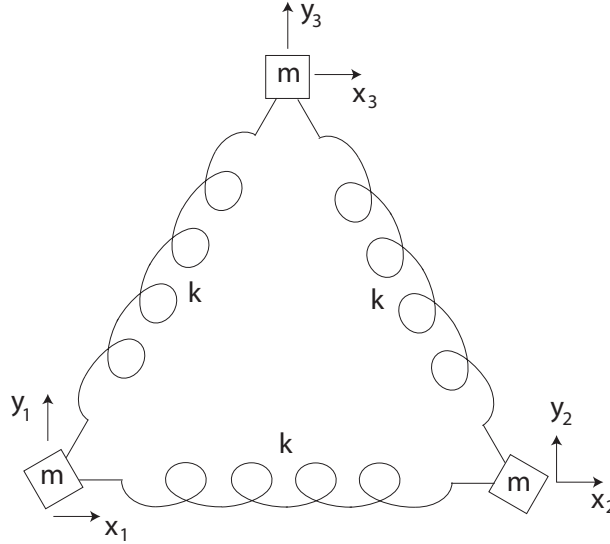


Figure 3.2: The three springs example, showing the coordinate system. Each coordinate pair has its origin at the center of its respective mass in the equilibrium position.

Finally, we use the first orthogonality relation to isolate a particular coefficient, obtaining,

$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} \chi^{(j)*}(C_k) \chi(C_k) N_k. \quad (3.91)$$

3.10 Example Application

For our first example of a physical application, we consider an arrangement of springs and masses which have a particular symmetry in the equilibrium position. We'll consider here the case of an equilateral triangle, expanding on the example in Mathews & Walker chapter 16.

Suppose that we have a system of three equal masses, m , located (in equilibrium) at the vertices of an equilateral triangle. The three masses are connected by three identical springs of strength k . See Fig. 3.2. The question we wish to answer is: If the system is constrained to move in a plane, what are the normal modes? We'll use group theory to analyze what happens when a normal mode is excited, potentially breaking the equilateral triangular symmetry to some lower symmetry.

Let the coordinates of each mass, relative to the equilibrium position, be $x_i, y_i, i = 1, 2, 3$. The state of the system is given by the 6-dimensional vector:

$\eta = (x_1, y_1, x_2, y_2, x_3, y_3)$, as a function of time. The kinetic energy is:

$$T = \frac{m}{2} \sum_{i=1}^6 \dot{\eta}_i^2. \quad (3.92)$$

Likewise, the potential energy, for small perturbations about equilibrium, is given by:

$$V = \frac{k}{2} \left\{ (x_2 - x_1)^2 + \left[-\frac{1}{2}(x_3 - x_2) + \frac{\sqrt{3}}{2}(y_3 - y_2) \right]^2 + \left[\frac{1}{2}(x_1 - x_3) + \frac{\sqrt{3}}{2}(y_1 - y_3) \right]^2 \right\}. \quad (3.93)$$

Or, we may write:

$$V = \frac{k}{2} \sum_{i,j=1}^6 U_{ij} \eta_i \eta_j, \quad (3.94)$$

where

$$U = \frac{1}{4} \begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix}. \quad (3.95)$$

The equations of motion ($F = ma$) are:

$$m\ddot{\eta}_i = -\frac{\partial V}{\partial \eta_i} = -k \sum_{j=1}^6 U_{ij} \eta_j. \quad (3.96)$$

In a normal mode,

$$\eta = \mathbf{A} e^{i\omega t}, \quad (3.97)$$

where \mathbf{A} is a constant 6-vector, and hence,

$$-m\omega^2 \eta_i = -k \sum_{j=1}^6 U_{ij} \eta_j, \quad (3.98)$$

or,

$$\sum_{j=1}^6 U_{ij} \eta_j = \lambda \eta_i, \quad \text{where } \lambda = \frac{m\omega^2}{k}. \quad (3.99)$$

That is, the normal modes are the eigenvectors of U , with frequencies given in terms of the eigenvalues. In principle, we need to solve the secular equation $|U - \lambda I| = 0$, a sixth-order polynomial equation, in order to get the eigenvalues. Let's see how group theory can help make this tractable, by incorporating the symmetry of the system.

Each eigenvector “generates” an irreducible representation when acted upon by elements of the symmetry group. Consider a coordinate system in which U is diagonal (such a coordinate system must exist, since U is Hermitian):

$$U = \begin{pmatrix} \lambda_a & & & & & \\ & \ddots & & & & \\ & & \lambda_a & & & \\ & & & \lambda_b & & \\ & & & & \ddots & \\ & & & & & \lambda_b \\ & & & & & & \ddots \end{pmatrix}, \quad (3.100)$$

where the first n_a coordinate vectors in this basis belong to eigenvalue λ_a , and transform among themselves according to irreducible representation $D^{(a)}$, and so forth.

What is the appropriate symmetry group? Well, it must be the group, C_{3v} , of operations which leaves an equilateral triangle invariant. This group is generated by taking products of a rotation by $2\pi/3$, which we will call R , and a reflection about the y -axis, which we will call P . The entire group is then given by the 6 elements $\{e, R, R^2, P, PR, PR^2\}$. Note that this group is isomorphic with the group of permutations of three objects, S_3 . The classes are:

$$\{e\}, \{R, R^2\}, \{P, PR, PR^2\}. \quad (3.101)$$

As there are three classes, there must be three irreducible representations, and hence their dimensions must be 1, 1, and 2. Thus, we can easily construct the character table in Table 3.3.

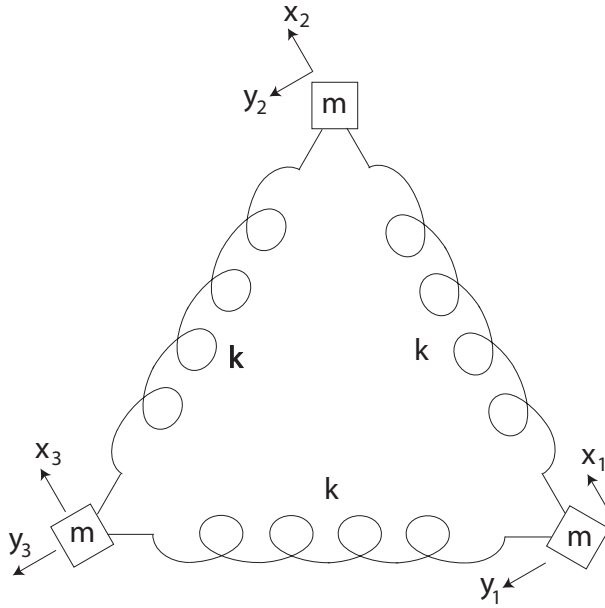
The first row is given by the dimensions of the irreps, since these are the traces of the identity matrices in those irreps. The first column is all ones, since this is the trivial irrep where every element of C_{3v} is represented by the number 1. The second and third row of the second column may be obtained by orthogonality with the first row (remembering the N_k weights), noticing that in a one-dimensional representation the traces are the same as the representation. In particular, the representation of R must be a cube root of one, and the representation of P must be a square root of one. Finally, the second and third rows of the final column are readily determined using the orthogonality relations. Note that in this example, we don't actually need to construct the non-trivial representations to determine the character table. In general, it may be necessary to construct a few of the matrices explicitly.

There is a 6-dimensional representation of C_{3v} which acts on our 6-dimensional coordinate space. We wish to decompose this representation into irreducible representations (why? because that will provide a breakdown of the normal modes by their symmetry under C_{3v}). It is sufficient to know the characters, which we obtain by explicitly considering the action of one element from each class.

Clearly, $\eta = D(e)\eta$, hence $D(e)$ is the 6×6 identity matrix. Its character is $\chi(C_1) = 6$.

Table 3.3: Character table for C_{3v} .

N_k	$l_i \rightarrow$ class \downarrow ; irrep \rightarrow	$l_i = 1$ $\chi^{(1)}$	$l_2 = 1$ $\chi^{(2)}$	$l_3 = 2$ $\chi^{(3)}$
1	$\{e\}$	1	1	2
2	$\{R, R^2\}$	1	1	-1
3	$\{P, PR, PR^2\}$	1	-1	0

Figure 3.3: The three springs example, showing result of a rotation by $2\pi/3$.

Now consider a rotation by $2\pi/3$, see Fig. 3.3. The 6×6 matrix representing this rotation is:

$$D(R) = \begin{pmatrix} 0 & 0 & r \\ r & 0 & 0 \\ 0 & r & 0 \end{pmatrix}, \quad (3.102)$$

where r is the 2×2 rotation matrix:

$$r = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \quad (3.103)$$

We see that the trace is zero, that is $\chi(C_2) = 0$.

The action of P is to interchange masses 1 and 2, and reflect the x coordi-

nates:

$$D(P) = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad (3.104)$$

where p is the 2×2 reflection matrix:

$$p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.105)$$

We see that the trace is again zero, that is $\chi(C_3) = 0$.

With these characters, we are now ready to decompose D into the irreps of C_{3v} . We wish to find the coefficients a_1, a_2, a_3 in:

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus a_3 D^{(3)}. \quad (3.106)$$

They are given by:

$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} N_k \chi^{(j)*}(C_k) \chi(C_k). \quad (3.107)$$

The result is:

$$\begin{aligned} a_1 &= \frac{1}{6}(1 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 0 + 3 \cdot 1 \cdot 0) = 1 \\ a_2 &= \frac{1}{6}(1 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 0 + 3 \cdot -1 \cdot 0) = 1 \\ a_3 &= \frac{1}{6}(1 \cdot 2 \cdot 6 + 2 \cdot -1 \cdot 0 + 3 \cdot 0 \cdot 0) = 2. \end{aligned} \quad (3.108)$$

That is,

$$D = D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)}. \quad (3.109)$$

In the basis corresponding to the eigenvalues we thus have:

$$U = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_{31} & & 0 & \\ & & & \lambda_{31} & & \\ & & 0 & & \lambda_{32} & \\ & & & & & \lambda_{32} \end{pmatrix}, \quad (3.110)$$

where λ_1 corresponds to $D^{(1)}$, λ_2 to $D^{(2)}$, and $\lambda_{31}, \lambda_{32}$ to two instances of $D^{(3)}$. Thus, we already know that there are no more than four distinct eigenvalues, that is, some of the six modes have the same frequency.

Let's see that we can find the actual frequencies without too much further work. Consider $D(g)U$ in this diagonal coordinate system. In this basis we must have:

$$D(g) = \begin{pmatrix} D^{(1)}(g) & & & & & \\ & D^{(2)}(g) & & & 0 & \\ & & 0 & & & \\ & & & D^{(3)}(g) & & \\ & & & & D^{(3)}(g) & \\ & & & & & D^{(3)}(g) \end{pmatrix}, \quad (3.111)$$

and hence,

$$D(g)U = \begin{pmatrix} \lambda_1 D^{(1)}(g) & & & & & \\ & \lambda_2 D^{(2)}(g) & & & & \\ & & 0 & & & \\ & & & \lambda_{31} D^{(3)}(g) & & \\ & & & & & \\ & & & & & \lambda_{32} D^{(3)}(g) \end{pmatrix}. \quad (3.112)$$

We don't know what this coordinate system is, but we may consider quantities which are independent of coordinate system, such as the trace:

$$\text{Tr}[D(g)U] = \lambda_1 \chi^{(1)}(g) + \lambda_2 \chi^{(2)}(g) + (\lambda_{31} + \lambda_{32}) \chi^{(3)}(g). \quad (3.113)$$

Referring to Eqn. 3.95 we find, for $g = e$:

$$\text{Tr}[D(e)U] = \text{Tr} U = \frac{1}{4}(5 + 3 + 5 + 3 + 2 + 6) = 6. \quad (3.114)$$

For $g = R$:

$$\begin{aligned} \text{Tr}[D(R)U] &= \text{Tr} \frac{1}{2} \begin{pmatrix} 0 & & & -1 & -\sqrt{3} \\ & 0 & & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & & & \\ \sqrt{3} & -1 & & & 0 \\ & & 0 & & \\ & & & -1 & -\sqrt{3} \\ & & & \sqrt{3} & -1 \\ & & & & 0 \end{pmatrix} \times \\ &\quad \frac{1}{4} \begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix} \\ &= \frac{1}{8}(1 + 3 - 3 + 3 + 4 + 0 + 1 - 3 + 3 + 3) = \frac{3}{2} \end{aligned} \quad (3.115)$$

For $g = P$:

$$\begin{aligned} \text{Tr}[D(P)U] &= \text{Tr} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix} \\ &= \frac{1}{4}(4 + 0 + 4 + 0 - 2 + 6) = 3. \end{aligned} \quad (3.116)$$

This gives us the three equations:

$$\begin{aligned} 6 &= \lambda_1 + \lambda_2 + 2(\lambda_{31} + \lambda_{32}) \\ \frac{3}{2} &= \lambda_1 + \lambda_2 - (\lambda_{31} + \lambda_{32}) \\ 3 &= \lambda_1 - \lambda_2. \end{aligned} \quad (3.117)$$

Hence,

$$\lambda_1 = 3 \quad (3.118)$$

$$\lambda_2 = 0 \quad (3.119)$$

$$\lambda_{31} + \lambda_{32} = \frac{3}{2}. \quad (3.120)$$

To determine λ_{31} and λ_{32} , we could consider another invariant, such as

$$\text{Tr}U^2 = \lambda_1^2 + \lambda_2^2 + 2(\lambda_{31}^2 + \lambda_{32}^2). \quad (3.121)$$

Alternatively, we may use some physical insight: There must be three degrees of freedom with eigenvalue 0, corresponding to an overall rotation of the system and overall translation of the system in two directions. Thus, choose $\lambda_{31} = 0$ and then $\lambda_{32} = 3/2$.

The frequencies are $\omega = \sqrt{\lambda k/m}$. The highest frequency is $\omega = \sqrt{3k/m}$, corresponding to the “breathing mode” in which the springs all expand or contract in unison. Note that this is the mode corresponding to the identity representation; the symmetry of the triangle is not broken in this mode.

3.11 Another example

Let us consider another simple example (again an expanded discussion of an example in Mathews & Walker, chapter 16), to try to get a more intuitive picture of the connection between eigenfunctions and irreducible representations:

Consider a square “drumhead”, and the connection of its vibrational modes with representations of the symmetry group of the square. We note that two eigenfunctions which are related by a symmetry of the square must have the same eigenvalue – otherwise this would not be a symmetry. The symmetry group of the square (see Fig. 3.4) is generated by a 4-fold axis, plus mirror planes joining the sides and vertices.

This group has the elements:

$$\{e, M_a, M_b, M_\alpha, M_\beta, R_{\pm\pi/2}, R_\pi\}. \quad (3.122)$$

Thus, the order is $h = 8$. The classes are readily seen to be:

$$\begin{aligned} C_1 &= \{e\} \\ C_2 &= \{M_a, M_b\} \\ C_3 &= \{M_\alpha, M_\beta\} \\ C_4 &= \{R_\pi\} \\ C_5 &= \{R_{\pi/2}, R_{-\pi/2}\} \end{aligned} \quad (3.123)$$

We must have $\sum_{i=1}^{n_r} \ell_i^2 = 8$, but $n_r = 5$, and therefore $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1$, and $\ell_5 = 2$ are the dimensions of the irreducible representations.

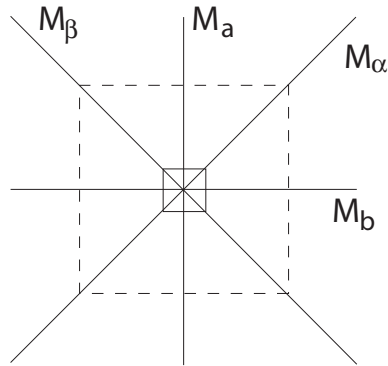


Figure 3.4: The symmetry group of the square.

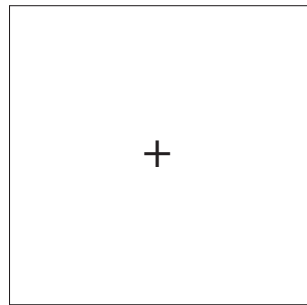


Figure 3.5: The lowest excitation of the square drumhead. The “plus” in the center is supposed to indicate that the whole drumhead is oscillating back and forth through the plane of the square.

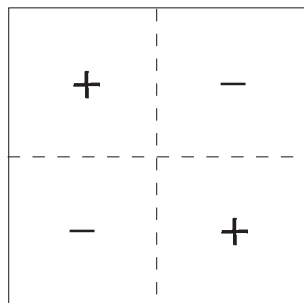


Figure 3.6: A higher excitation of the square drumhead.

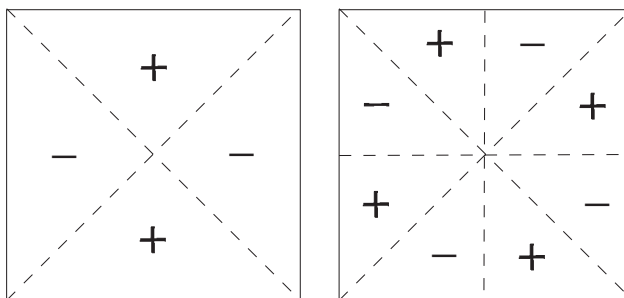


Figure 3.7: Two additional excitations of the square drumhead, corresponding to one-dimensional representations.

Let us consider some vibrational modes and see what representations they generate: The lowest mode is just when the whole drumhead vibrates back and forth, Fig. 3.5.

The action of any group element on this mode is to transform it into itself, hence, this mode generates the trivial representation where all elements are represented by the number 1.

Another mode is shown in Fig. 3.6.

This mode is also non-degenerate, hence it must generate also a one-dimensional representation, but it is no longer the trivial representation, since it is not invariant under the action of all of the elements of the group. For example, $R_{\pi/2}$ yields a minus sign on this mode.

Likewise, the modes shown in Fig. 3.7 are non-degenerate and generate new one-dimensional irreducible representations. It may be seen that these one-dimensional irreps are all inequivalent, as the actions of the group elements differ in the different irreps.

Finally, we have the degenerate modes illustrated in Fig. 3.8. These two modes transform among themselves under the group operations, hence generate a two-dimensional irreducible representation.

We might wonder about the modes illustrated in Fig. 3.9. These also gen-

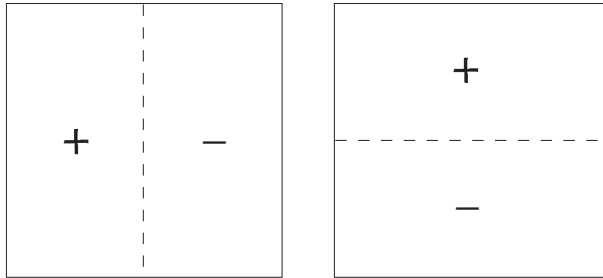


Figure 3.8: Two degenerate modes, generating a two-dimensional representation.

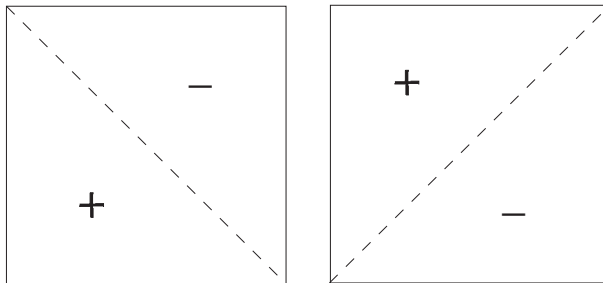


Figure 3.9: Another pair of degenerate modes, generating a two-dimensional representation.

erate a two-dimensional irrep. However, the reader is encouraged to show that this irrep is equivalent to the one above.

We might also wonder about the modes illustrated in Fig. 3.10. These also generate a two-dimensional representation. In this case, however, the representation is reducible.

There are also modes which generate the same irreps already considered, but corresponding to higher excitations. For example, see Fig. 3.11

3.12 Direct Product Theory

The direct product of two matrices A and B is the set of product elements obtained by multiplying every element of A by every element of B . It is convenient to think of these products as arranged in a “direct product matrix” form. For example, if A is $n \times n$, and B is $m \times m$, the direct product matrix is $nm \times nm$. The multiplication of direct product matrices is defined so that they can describe successive transformations in a “product” space. A product space is formed out of two spaces so that a transformation in the product space is a combination of transformations done separately in each of the ordinary spaces such that the rule of successive transformations is obeyed in each of the ordinary

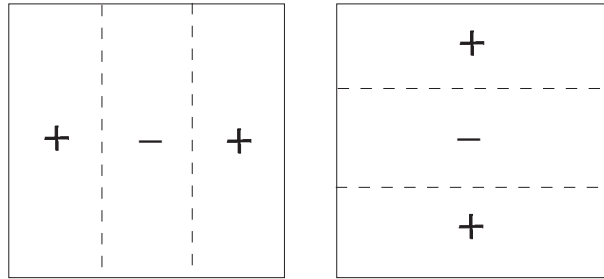


Figure 3.10: Another pair of degenerate modes, generating a two-dimensional representation, but this time a reducible representation.

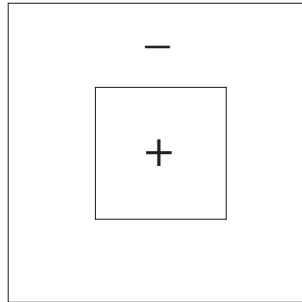


Figure 3.11: A higher excitation, generating the identity representation.

spaces separately.

Thus, if A, A' are operators in space a , and B, B' are operators in space b , then $A'' = AA'$ is the successive operation of A' , then A in space a , and $B'' = BB'$ is the successive operation of B' , then B in space b . Suppose we define “direct product” operators $C = A \otimes B$, and $C' = A' \otimes B'$. Then we require that

$$\begin{aligned} C'' &= CC' \\ &= (A \otimes B)(A' \otimes B') \\ &= AA' \otimes BB' \\ &= A'' \otimes B'' \end{aligned} \tag{3.124}$$

By components, this is:

$$\begin{aligned} C''_{ik,jm} &= \sum_{p,q} C_{ik,pq} C'_{pq,jm} \\ &= \sum_{p,q} A_{ip} B_{kq} A'_{pj} B'_{qm}, \quad \text{since...} \\ &= \sum_{p,q} A_{ip} A'_{pj} B_{kq} B'_{qm} \\ &= A''_{ij} B''_{km}. \end{aligned} \tag{3.125}$$

One possible way to write out the direct product matrix is:

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.126}$$

If we are given two groups, $G_a = (\{a_i\}, \circ)$ of order h_a and $G_b = (\{b_i\}, *)$ of order h_b , the *direct product group*, $G_a \otimes G_b$, is formed by the elements consisting of all ordered pairs (a_i, b_j) with multiplication defined by:

$$(a_i, b_j)(a_k, b_\ell) \equiv (a_i \circ a_k, b_j * b_\ell). \tag{3.127}$$

As usual, we typically drop the explicit operation symbols in the hopes that the appropriate operation is understood from context. The reader is urged to demonstrate that we have in fact defined a group here.

We list some facts concerning direct product groups:

1. In the groups $G_a = \{a_i\}$ and $G_b = \{b_i\}$, the indices i and j run over some index sets, not necessarily finite or even countable. The order of $G_a \otimes G_b$ is the product of the orders of the two groups, i.e., $h_a h_b$. This may be infinite.
2. If e_a is the identity element of G_a and e_b is the identity element of G_b , then the set of elements in $G_a \otimes G_b$ of the form (e_a, b_i) yields a subgroup isomorphic with G_b , and those of the form (a_i, e_b) yield a subgroup isomorphic with G_a .

Table 3.4: Character table for C_{3v} .

N_k	$\ell_i \rightarrow$ class \downarrow ; irrep \rightarrow	$\ell_i = 1$ $\chi^{(1)}$	$\ell_2 = 1$ $\chi^{(2)}$	$\ell_3 = 2$ $\chi^{(3)}$
1	$\{e\}$	1	1	2
2	$\{R, R^2\}$	1	1	-1
3	$\{P, PR, PR^2\}$	1	-1	0

Table 3.5: Character table for the inversion group, \mathcal{I} .

N_k	$\ell_i \rightarrow$ class \downarrow ; irrep \rightarrow	$\ell_i = 1$ $\chi^{(1)}$	$\ell_2 = 1$ $\chi^{(2)}$
1	$\{e\}$	1	1
1	$\{i\}$	1	-1

- The classes of the direct product group are given by the direct products of the classes of the original groups.
- The direct products of the matrices representing G_a and G_b provide representations of $G_a \otimes G_b$ under the matrix multiplication rule for direct product matrices.
- If $D_a^{(i)}(a_r)$ and $D_b^{(j)}(b_s)$ are irreps of G_a and G_b , respectively, then

$$D_c^{(ij)}(c_{rs}) \equiv D_a^{(i)}(a_r) \otimes D_b^{(j)}(b_s) \quad (3.128)$$

is an irrep of $G_c = G_a \otimes G_b$. Furthermore, there are no additional irreps besides those constructed in this way. Note that this, plus the previous item on representations, implies that the character table of the product group is:

$$\chi_c^{(ij)}(c_{rs}) = \chi_a^{(i)}(a_r) \chi_b^{(j)}(b_s). \quad (3.129)$$

Let's look at an example of the construction of a character table for a direct product group. Suppose we have the symmetry group C_{3v} of the equilateral triangle. We have already obtained the character table for this group in our example on springs in Section 3.10. This table is repeated in Table 3.4. Recall that R stands for a rotation by $2\pi/3$, and P is one of the mirrors containing the rotation axis and a vertex.

Now suppose that we wish to add to this group the operation of inversion. The resulting group is called D_{3d} . The inversion group, \mathcal{I} , is a two-element group, consisting of the identity e and the inversion operator i . The only possible character table for a group of order two is shown in Table 3.5.

We wish to obtain the character table for the product group $D_{3d} = C_{3v} \otimes \mathcal{I}$. Recall that the character of a representation is the trace of a matrix, so we must

Table 3.6: Character table for D_{3d} .

N_k	$\ell_i \rightarrow$ class \downarrow ; irrep \rightarrow	$\ell_1 = 1$ $\chi^{(1)}$	$\ell_2 = 1$ $\chi^{(2)}$	$\ell_3 = 2$ $\chi^{(3)}$	$\ell_4 = 1$ $\chi^{(4)}$	$\ell_5 = 1$ $\chi^{(5)}$	$\ell_6 = 2$ $\chi^{(6)}$
1	$C_1 = \{e\}$	1	1	2	1	1	2
2	$C_2 = \{R, R^2\}$	1	1	-1	1	1	-1
3	$C_3 = \{P, PR, PR^2\}$	1	-1	0	1	-1	0
1	$C_4 = \{ie\}$	1	1	2	-1	-1	-2
2	$C_5 = \{iR, iR^2\}$	1	1	-1	-1	-1	1
3	$C_6 = \{iP, iPR, iPR^2\}$	1	-1	0	-1	1	0

determine the trace of a direct product matrix. If $c = a \otimes b$ is the matrix direct product of matrices a and b , then

$$\begin{aligned}
 \chi(c) &= \chi(a \otimes b) \\
 &= \sum_{k\ell} (a \otimes b)_{k\ell, k\ell} \\
 &= \sum_k a_{kk} \sum_\ell b_{\ell\ell} \\
 &= \chi(a)\chi(b).
 \end{aligned} \tag{3.130}$$

There will be $2 \times 3 = 6$ irreps for our product group (we have doubled the number of classes of D_3 . The character table must be as shown in Table 3.6.

The order of D_{3d} is $h = 12$, which agrees with the sum of the squares of the dimensions of the irreps $\ell_k = 1, 1, 1, 1, 2, 2$. We remark also that the character table looks like the direct product of the input character tables:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{3.131}$$

Expressing a group as a direct product of smaller groups provides a useful method for studying the irreps of the larger group. Note that $\mathcal{I} = \{e, i\}$ is an abelian invariant subgroup of D_{3d} ($gag^{-1} \in \mathcal{I}$, $\forall a \in \mathcal{I}$ and $\forall g \in D_{3d}$). Therefore, D_{3d} is not a simple group (since it contains a proper invariant subgroup), nor is it semi-simple (since the invariant subgroup is abelian).

We may write the suggestive notation $C_{3v} = D_{3d}/\mathcal{I}$ and refer to C_{3v} as the “factor group” or “quotient group”. Since C_{3v} is the group that leaves the triangle invariant, we refer to it as the “little group” of D_{3d} (or the “little group of the triangle”).

3.13 Generating Additional Representations

Given one or more representations D of a group G , there are various ways of generating additional representations. We have already seen the direct sum method. Let us now see some others. Relax, for now, the assumption of unitary representations. First, there are three simple operations on representation D which will give us (possibly) new representations of the same dimension:

1. *Adjoint Representation:* Given a group G with a (invertible) representation $D(a), a \in G$, consider the set of matrices obtained by taking the inverse transpose of $D(a)$: $[D(a)^{-1}]^T$. This is also a representation of G , since,

$$\begin{aligned} [D(ab)^{-1}]^T &= [D(b)^{-1}D(a)^{-1}]^T \\ &= [D(a)^{-1}]^T[D(b)^{-1}]^T, \end{aligned}$$

hence, the multiplication table is preserved. This is called the “adjoint representation”, \bar{D} .

2. *Complex Conjugate Representation:* Given a representation D , consider the matrices formed by taking the complex conjugate of the elements of $D(a)$: $[D(a)]^*$. We have,

$$\begin{aligned} D(ab)^* &= [D(a)D(b)]^* \\ &= D(a)^*D(b)^*, \end{aligned}$$

so this also defines a representation. It is called the “complex conjugate representation”, D^* .

3. Finally, we also obtain a representation by taking the complex conjugate of the adjoint representation:

$$[D(a)^{-1}]^\dagger = \bar{D}(a)^*. \quad (3.132)$$

We note that $D, \bar{D}, D^*, \bar{D}^*$ are all either reducible or irreducible representations, which may, or may not, be equivalent. Thus, this is one thing to try towards finding new (irreducible) representations for G . Note that if we have a unitary representation, which is always possible for a finite group,

$$\bar{D}(g) = [D(g)^{-1}]^T = [D(g)^\dagger]^T = D^*(g), \quad (3.133)$$

hence, the adjoint representation is identical with the complex conjugate representation.

If the representation is real, then $D(g) = D(g)^*$, and $\chi(g)$ is real. If, instead, we know that $\chi(g)$ is real, then $\chi(g) = \text{Tr}[D(g)] = \text{Tr}[D(g)^*]$, and therefore D and D^* are equivalent. If, on the other hand, $\chi(g)$ is complex, then D and D^* are not equivalent representations. We can state these observations in the form of a theorem:

Theorem: D and D^* are equivalent representations if and only if their characters are real.

Let us revisit briefly our general orthogonality relation. Since we stated it for finite groups, we have been justified in assuming we can always deal with unitary representations. However, we might happen to deal at some point with a non-unitary representation. In general, the orthogonality relation for irreducible representations $D^{(i)}$, and $D^{(j)}$ reads:

$$\sum_{g \in G} D^{(i)}(g)_{\mu\nu} D^{(j)}(g^{-1})_{\alpha\beta} = \frac{h}{\ell_i} \delta_{ij} \delta_{\mu\beta} \delta_{\nu\alpha}. \quad (3.134)$$

In terms of this general relation, we can repeat the earlier derivation of the “first orthogonality relation”, setting $\nu = \mu$, $\beta = \alpha$, and summing over μ and α , to obtain:

$$\sum_{g \in G} \chi^{(i)}(g) \chi^{(j)}(g^{-1}) = h \delta_{ij}. \quad (3.135)$$

But, in the adjoint representation, $\bar{D}(g) = [D(g)^{-1}]^T = D(g^{-1})^T$, and hence $\bar{\chi}(g) = \chi(g^{-1})$. Therefore,

$$\sum_{g \in G} \chi^{(i)}(g) \bar{\chi}^{(j)}(g) = h \delta_{ij}, \quad (3.136)$$

or, in terms of classes:

$$\sum_{k=1}^{n_c} N_k \chi^{(i)}(C_k) \bar{\chi}^{(j)}(C_k) = h \delta_{ij}. \quad (3.137)$$

Likewise, our second orthogonality relation in general is:

$$\sum_{i=1}^{n_r} \chi^{(i)}(C_k) \chi^{(i)}(C_\ell^{-1}) = \frac{h}{N_k} \delta_{k\ell}, \quad (3.138)$$

where C_ℓ^{-1} means take the inverses of the elements in class C_ℓ . Or, using again $\bar{\chi}(g) = \chi(g^{-1})$

$$\sum_{i=1}^{n_r} \chi^{(i)}(C_k) \bar{\chi}^{(i)}(C_\ell) = \frac{h}{N_k} \delta_{k\ell}, \quad (3.139)$$

With these forms, the expansion coefficients for the decomposition of an arbitrary representation, D , into irreps become:

$$a_m = \frac{1}{h} \sum_{k=1}^{n_c} N_k \chi(C_k) \bar{\chi}^{(m)}(C_k). \quad (3.140)$$

3.14 Kronecker Products and Clebsch-Gordan Coefficients

Given two representations $D^{(i)}$ and $D^{(j)}$ of a group G , we may construct a new representation, $D^{(i \times j)}$, by taking the direct product matrices:

$$D^{(i \times j)}(g) = D^{(i)}(g) \otimes D^{(j)}(g). \quad (3.141)$$

In components:

$$D^{(i \times j)}(g)_{\alpha\beta, \mu\nu} = D^{(i)}(g)_{\alpha\mu} D^{(j)}(g)_{\beta\nu}. \quad (3.142)$$

It is left to the reader to verify that $D^{(i \times j)}$ is in fact a representation for G . It is called a *product representation* or a *Kronecker product*. As with direct product groups, we find:

$$\chi^{(i \times j)}(g) = \chi^{(i)}(g) \chi^{(j)}(g). \quad (3.143)$$

Let us now assume that $D^{(i)}$ and $D^{(j)}$ are irreducible representations. The product representation $D^{(i \times j)}$ may, however, be reducible. We would like to find the decomposition into irreps:

$$D^{(i \times j)} = a_1 D^{(1)} \oplus \dots \oplus a_{n_r} D^{(n_r)}. \quad (3.144)$$

This reduction is called the *Clebsch-Gordan series*. For the coefficients, we have:

$$\begin{aligned} a_m &= \frac{1}{h} \sum_{k=1}^{n_c} N_k \chi^{(i \times j)}(C_k) \bar{\chi}^{(m)}(C_k) \\ &= \frac{1}{h} \sum_{g \in G} \chi^{(i \times j)}(g) \bar{\chi}^{(m)}(g) \\ &= \frac{1}{h} \sum_{g \in G} \chi^{(i)}(g) \chi^{(j)}(g) \bar{\chi}^{(m)}(g). \end{aligned} \quad (3.145)$$

For example, in the note on rotations in quantum mechanics (section 10), the Clebsch-Gordan series for the group $SU(2)$ (an isomorphic representation of the rotation group in quantum mechanics) is obtained:

$$D^{(i \times j)} = \bigoplus_{m=|i-j|}^{i+j} D^{(m)}. \quad (3.146)$$

We'll proceed now to define the notion of "Clebsch-Gordan coefficients" (note that the a_m coefficients in the reduction above are sometimes referred to as Clebsch-Gordan coefficients; this will not be our usage). We start by expressing the Clebsch-Gordan series in a different notation:

$$D^{(i \times j)} = \bigoplus_{m=1}^{n_r} a_m D^{(m)} = \bigoplus_{m=1}^{n_r} (ijm) D^{(m)}. \quad (3.147)$$

3.14. KRONECKER PRODUCTS AND CLEBSCH-GORDAN COEFFICIENTS 57

That is, $(ijm) \equiv a_m$, and our coefficient names now contain explicitly the information of which product representation we are looking at. Notice that there is a symmetry, $(ijm) = (jim)$, since $D^{(m)}$ will appear the same number of times in $D^{(i \times j)}$ as in $D^{(j \times i)}$.

For physical applications (e.g., quantum mechanical angular momentum), we are especially interested in determining the basis functions for the representations in the Kronecker product. For the irrep $D^{(i)}$ (that is, for the vector space acted upon by this representation) we have the basis functions:

$$\{\psi_\alpha^{(i)}; \alpha = 1, 2, \dots, \ell_i\}, \quad (3.148)$$

where ℓ_i is the dimension of irrep i . Likewise, for irrep $D^{(j)}$ we have basis functions:

$$\{\phi_\beta^{(j)}; \beta = 1, 2, \dots, \ell_j\}. \quad (3.149)$$

Since we are considering the product representation $D^{(i)} \otimes D^{(j)}$ we may ask for the ℓ_m functions

$$\{\omega_\gamma^{(m)}; \gamma = 1, \dots, \ell_m\} \quad (3.150)$$

that are linear combinations of the products $\psi_\alpha^{(i)}$ and $\phi_\beta^{(j)}$, and which form a basis for the irrep $D^{(m)}$. Such a set of functions $\{\omega_\gamma^{(m)}\}$ exists only if $D^{(m)}$ is contained in $D^{(i)} \otimes D^{(j)}$, that is, only if $(ijm) > 0$.

Now, (ijm) may be one, but $(ijm) > 1$ is also possible, in which case there will be more than one such sets of functions. In general, there will be precisely (ijm) independent sets of functions $\{\omega_\gamma^{(m)}\}$ formed from the products $\psi_\alpha^{(i)} \phi_\beta^{(j)}$. We'll label them:

$$\{\omega_\gamma^{(m\tau_m)}; \tau_m = 1, \dots, (ijm)\} \quad (3.151)$$

More explicitly, these are functions of the form:

$$\omega_\gamma^{(m\tau_m)} = \sum_{\alpha=1}^{\ell_i} \sum_{\beta=1}^{\ell_j} \psi_\alpha^{(i)} \phi_\beta^{(j)} (i\alpha, j\beta | m\tau_m \gamma) \quad (3.152)$$

The quantities $(i\alpha, j\beta | m\tau_m \gamma)$ are called *Clebsch-Gordan coefficients*.

It is important to understand that all we are really doing here is describing a transformation of basis between alternative bases in the space operated on by the product representation. We remark also that in the case of the quantum mechanical rotation group, the numbers τ_m are never greater than one - the rotation group is said to be *simply reducible*.

The total number of functions $\omega_\gamma^{(m\tau_m)}$ must be the same as the total number of product functions $\psi_\alpha^{(i)} \phi_\beta^{(j)}$:

$$\sum_{m=1}^{n_r} (ijm) \ell_m = \ell_i \ell_j. \quad (3.153)$$

Hence, the Clebsch-Gordan coefficients make a $\ell_i \ell_j \times \ell_i \ell_j$ matrix. As our expansion for $\omega_\gamma^{(m\tau_m)}$ is just a basis transformation, we can write the inverse transformation:

$$\psi_\alpha^{(i)} \phi_\beta^{(j)} = \sum_{\gamma, m, \tau_m} (m\tau_m \gamma | i\alpha, j\beta) \omega_\gamma^{(m\tau_m)}. \quad (3.154)$$

Substituting back into the original equation (3.152), we find:

$$\omega_\gamma^{(m\tau_m)} = \sum_{\alpha, \beta} \sum_{\gamma', m', \tau'_m} (m'\tau'_m \gamma' | i\alpha, j\beta) (i\alpha, j\beta | m\tau_m \gamma) \omega_{\gamma'}^{(m'\tau'_m)}, \quad (3.155)$$

or,

$$\sum_{\alpha, \beta} (m'\tau'_m \gamma' | i\alpha, j\beta) (i\alpha, j\beta | m\tau_m \gamma) = \delta_{mm'} \delta_{\tau_m \tau'_m} \delta_{\gamma \gamma'}. \quad (3.156)$$

Alternatively, substituting Eqn. 3.152 into Eqn. 3.154, we obtain:

$$\sum_{m, \tau_m, \gamma} (i\alpha', j\beta' | m\tau_m \gamma) (m\tau_m \gamma | i\alpha, j\beta) = \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \quad (3.157)$$

At least for unitary representations, it may be shown that the matrix of Clebsch-Gordan coefficients is a matrix which puts $D^{(i)} \times D^{(j)}$ into reduced form.

3.15 Angular Momentum in Quantum Mechanics

The theory of angular momentum in quantum mechanics is developed in detail in the note on this subject linked to the Ph 129 page. Here, we'll summarize a few of the key elements relative to our discussion of group theory. As the rotation group is an infinite group, we'll also remark on the extension of our discussion to infinite groups.

As an explicit function, the spherical harmonic $Y_{\ell m}$ is given by:

$$Y_{\ell m}(\theta, \phi) = \frac{(-1)^m}{2^\ell \ell!} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell+m)!}{(\ell-m)!}} e^{im\phi} \frac{1}{(\sin \theta)^m} \left(\frac{d}{d \cos \theta} \right)^{\ell+m} (1 - \cos^2 \theta)^\ell, \quad (3.158)$$

where $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. However, it is perhaps more profound to define the $Y_{\ell m}$ in terms of the matrices $D^\ell(R)$ which give the irreducible representations of the rotation group.

Consider the rotation R expressed in terms of the Euler angles α, β, γ :

$$R = R(\alpha, \beta, \gamma) = R_z(\gamma) R_y(\beta) R_z(\alpha). \quad (3.159)$$

Choosing $\alpha = 0$, a vector along the z -axis may be rotated to $\theta = \beta$ and $\phi = \gamma$. We consider the rotation matrices with components:

$$\begin{aligned} D^\ell(\gamma, \beta, \alpha)_{mm'} &= e^{-i(m\gamma+m'\alpha)} d_{mm'}^\ell(\beta) \\ &= \langle \ell m | D^\ell(R) | \ell, m' \rangle, \end{aligned} \quad (3.160)$$

with the interpretation that these are the rotation matrices acting on a vector space of functions corresponding to angular momentum ℓ . We then define the spherical harmonics:

$$Y_{\ell m}(\theta, \phi) \equiv \sqrt{\frac{2\ell+1}{4\pi}} D_{m0}^{*\ell}(\phi, \theta, 0). \quad (3.161)$$

Note that since $m' = 0$, the spherical harmonics describe states with integer angular momentum only.

As mentioned before, the Clebsch-Gordan coefficients describe a change of basis. Consider a system of two “particles”, with spins j_1 and j_2 . We may describe the (angular momentum) state of these particles according to:

$$|j_1 m_1 j_2 m_2\rangle, \quad (3.162)$$

where m_1 and m_2 are the z -components of the spins. However, we might also describe the state by specifying the total angular momentum, j , and the total component along the z -axis, m ($= m_1 + m_2$):

$$|j_1 j_2 j m\rangle. \quad (3.163)$$

This situation corresponds to the reduction of a product representation $D^{(j_1)} \otimes D^{(j_2)}$ into irreducible representations $D^{(j)}$. The Clebsch-Gordan coefficients (also known in this context as vector addition or Wigner coefficients) merely tell us how to transform from one basis to the other. For example,

$$|j_1 j_2 j m\rangle = \sum_{m_1, m_2} (j_1 m_1, j_2 m_2 | j m) |j_1 m_1 j_2 m_2\rangle, \quad (3.164)$$

where we have omitted the $\tau_j = 1$.

As mentioned already, we may show that for the rotation group ($SU(2)$) the Clebsch-Gordan series is

$$D^{(j_1 \times j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)}. \quad (3.165)$$

The proof of this relies on

$$\chi^{(j_1 \times j_2)} = \chi^{(j_1)} \chi^{(j_2)}, \quad (3.166)$$

and on the orthogonality relations:

$$\frac{1}{h} \sum_{g \in G} \chi^{(i)}(g) \bar{\chi}^{(j)}(g) = \delta_{ij}. \quad (3.167)$$

But this is an infinite group, $h = \infty$. We are faced with the issue of defining “ $\frac{1}{h} \sum_{g \in G}$ ”.

To generalize this to the case of an infinite group, notice that $\frac{1}{h} \sum_{g \in G}$ is a kind of average over the elements of the group. Since the rotation group depends

on three continuously varying parameters, we expect our sum to become some kind of integral. The question in constructing the appropriate integral is how to weight the various regions in parameter space. That is, we need to define a notion of the size of a set of rotations; we need a measure function, $\mu(\{R\})$, where $\{R\}$ is some set of rotations.

The measure function must satisfy the property that no rotation gets any bigger weight than any other. Consider a set $\{R\}$ of rotations. We can obtain another set of rotations, $\{R_0R\}$ by applying a specified rotation, R_0 , to each element of this set. The rotated set should be the same size as the original set. We require the following invariance of the measure (considering rotation group $O^+(3)$):

$$\mu(\{R\}) = \mu(\{R_0R\}), \quad \forall R_0 \in O^+(3). \quad (3.168)$$

For $O^+(3)$, the invariant measure, normalized such that the integral over the set of all rotations is one, is:

$$\mu(dR) = \frac{1}{8\pi^2} \sin \theta d\theta d\psi d\phi, \quad (3.169)$$

for rotations parameterized by the Euler angles:

$$\begin{aligned} 0 &\leq \psi < 2\pi \\ 0 &\leq \theta \leq \pi \\ 0 &\leq \phi < 2\pi. \end{aligned} \quad (3.170)$$

$$(3.171)$$

For $SU(2)$, the range of ϕ becomes $0 \leq \phi < 4\pi$, and the normalized invariant measure is:

$$\mu(dR) = \frac{1}{16\pi^2} \sin \theta d\theta d\psi d\phi. \quad (3.172)$$

See the note on angular momentum in quantum mechanics for a more detailed discussion.

In general, we may define such a measure on a group. If the group is “compact” (e.g., the rotation group is compact because it may be parameterized by parameters on a compact set), then it is possible to define a measure such that the measure $\mu(G)$ over the entire group is finite. In this case, many of our proofs may be readily modified to apply to the infinite groups. For example, every representation of a compact group is equivalent to a unitary representation. The rearrangement lemma, so handy in several of our proofs, reads,

$$\int_G f(u)\mu(du) = \int_G f(uv)\mu(du), \quad \forall v \in G. \quad (3.173)$$

In the case of $SU(2)$, this is:

$$\begin{aligned} \frac{1}{16\pi^2} \int_0^{2\pi} d\psi \int_{-1}^1 d \cos \theta \int_0^{4\pi} d\phi f [R(\psi, \theta, \phi)] = \\ \frac{1}{16\pi^2} \int_0^{2\pi} d\psi \int_{-1}^1 d \cos \theta \int_0^{4\pi} d\phi f [R(\psi_0, \theta_0, \phi_0)R(\psi, \theta, \phi)]. \end{aligned} \quad (3.174)$$

The invariant measure on a group is referred to as the *Haar measure*.

We'll expand on these notions in our note on Lie groups.

3.16 Exercises

1. For the Poincare group \bar{L} , show that any element $\Lambda(M, z)$ can be written as a product of a pure homogeneous transformation followed by a pure translation. Also show that it can be written as a pure translation followed by a pure homogeneous transformation.
2. Show that the object $\{x, y\}$ defined in Eqn. 3.36 is a scalar product.
3. Carry out the steps to demonstrate the decomposition of a representation into irreps,

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus \dots \oplus a_{n_r} D^{(n_r)}, \quad (3.175)$$

with coefficients:

$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} \chi^{(j)*}(C_k) \chi(C_k) N_k. \quad (3.176)$$

4. Derive the constant in Eqn. 3.84, that is, determine λ in:

$$\sum_{g \in C_k} D^{(i)}(g) = \lambda I. \quad (3.177)$$

5. Show that the two irreps generated according to Figs. 3.8 and 3.9 are equivalent.
6. Consider the group of all rotations in two dimensions: $G = \{R(\theta) : 0 \leq \theta < 2\pi\}$. As a linear operator on vectors in a two-dimensional Euclidean space, the elements of G may be represented (faithfully, or isomorphically) by the set of 2×2 matrices of the form:

$$D(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (3.178)$$

Show that this group can be decomposed into two one-dimensional representations, i.e., that you can find a transformation such that every element of G can be represented in the form:

$$D(\theta) = \begin{pmatrix} f(\theta) & 0 \\ 0 & g(\theta) \end{pmatrix}, \quad (3.179)$$

where the new representation is still faithful. You should find explicit expressions for f and g .

Hint: You want to find a similarity transformation, which just corresponds to a change in basis. You might consider the basis transformation so

often encountered in quantum mechanics and in optics, corresponding to describing states in terms of “circular polarization” instead of “linear polarization”.

7. Consider the dihedral group D_3 , which is isomorphic with the group of permutations of three objects, S_3 . Let V_2 be a two-dimensional Euclidean space spanned by orthonormal vectors $\mathbf{e}_x, \mathbf{e}_y$. Give the representation, D , of the elements of D_3 with respect to this basis. That is, express the transformed vectors $D(g)\mathbf{e}_i$ in terms of the original basis, and hence obtain representation D .
8. Consider the symmetry group, C_{4v} , of the square, consisting of rotations about the axis perpendicular to the square, and reflections about the vertical, horizontal, and diagonal axes in the plane of the square (but no mirror plane in the plane of the square).
 - (a) Construct a suitable set of irreducible representations of C_{4v} . That is, up to equivalence, construct all of the irreducible representations of this group.
 - (b) Give the character table for C_{4v} .
9. In problem 6 you consider the reducibility of a two-dimensional representation of the group of rotations in two dimensions. We may remark that this is an abelian group. Let us generalize that result: Consider a group, G , with a unitary representation D , consisting of unitary matrices $D(g)$, $g \in G$. If G is an abelian group, show that any such representation is, by a similarity transformation, equivalent to a representation by diagonal matrices (i.e., by a direct sum of 1×1 matrices). Note that we have already used group theory (Schur’s lemma) to argue the truth of this. In this problem, I want you to use what you know about matrix theory to demonstrate the result.
10. Construct the character table for the tetrahedral symmetry group T_d . You may wish to keep a copy of your result for problem 12.
11. Let’s take a peek at the relation of irreducible representations and the invariant subspaces of a vector space: Let V be the 6-dimensional function space consisting of polynomials of degree 2 in the two real variables x and y :

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h, \quad (3.180)$$

where a, b, c, d, e, h are complex numbers. If x, y transforms under the dihedral group D_3 (problem 7) as the coordinates of a 2-vector, then we obtain a 6-dimensional representation of D_3 on V . Identify the invariant subspaces of V under D_3 , and the corresponding irreducible representations contained in this six dimensional representation (don’t be afraid to use your intuition to make sure that what you find is sensible).

12. At last we are ready for a real physics application of group theory. We looked at the example of masses joined by springs in the shape of an equilateral triangle in this note. Now, let us consider the problem of four masses joined by springs. The four masses are at the corners of a tetrahedron, and the springs form the edges of the tetrahedron. Thus, there are six springs connecting the four masses. All four masses are equal, and all six springs are identical.

We wish to determine the frequencies of the normal modes for this system. Notice that to solve the secular equation, $|V - \lambda I| = 0$, presents a formidable image. A little physical intuition can reduce it somewhat, but it would take real cleverness to solve it completely. This cleverness comes in the form of group theory! Group theory permits one to incorporate in a systematic and deliberate way everything we know about the symmetry of the problem, hence reducing it to a simpler problem.

The problem is still not trivial – you should spend time thinking about convenient approaches in setting things up, and about ways to avoid doing unnecessary work. Above all, be careful, and check your results as you proceed. You already obtained the character table for the tetrahedral symmetry group in problem 10. This problem takes you the rest of the way through solving for the frequencies of the normal modes.

- (a) First step: Set up a 12-dimensional vector (coordinate system) describing the system, and derive the equations of motion, arriving finally at a set of linear equations that could be solved, in principle, to yield the frequencies of the normal modes.
 - (b) Second step: Obtain the character table for the twelve-dimensional representation of the tetrahedral symmetry group that acts on your 12-dimensional vector describing the system. Decompose this representation into irreducible representations.
 - (c) Final step: Obtain a small number of trace equations which you can use to solve to obtain the frequencies of the normal modes. Give the frequencies of the normal modes, and their degeneracies. Do your answers make physical sense?
13. The “quaternion” group consists of eight elements,

$$Q = \{1, -1, i, -i, j, -j, k, -k\}, \quad (3.181)$$

with multiplication table defined by (q is any element of Q):

$$\begin{aligned} 1q &= q \\ (-1)^2 &= 1 \\ (-1)q = q(-1) &= -q \\ i^2 = j^2 = k^2 = ijk &= -1 \end{aligned} \quad (3.182)$$

Find the character table for this group. Compare this character table with the character table for dihedral group D_4 . Are these two groups isomorphic?

14. As a follow-on to the drumhead example in this note, consider the symmetry group of the regular pentagon, as given by a five-fold axis and several mirror planes. Do not include the mirror plan containing the plane of the pentagon itself (although you may amuse yourself by considering what happens if you add this operation).
- List the group elements. Denote rotations with R 's, and mirror operations with M 's. Draw a picture! List the classes.
 - Construct the character table for the irreducible representations of this group.
 - Consider the mode of oscillation of a pentagonal drumhead where a nodal line extends from a vertex to the midpoint of the opposite side. Define (with pictures) a basis for the space generated by this mode and its degenerate partners. Give an explicit matrix for one element of each class of the group for the representation of the pentagonal symmetry group that is generated by these degenerate modes.
 - Decompose the representation found in part (c) into irreducible representations.
15. We would like to consider the (qualitative) effects on the energy levels of an atom which is moved from freedom to an external potential (a crystal, say) with cubic symmetry. Let us consider a one-electron atom and ignore spin for simplicity. Recall that the wave function for the case of the free atom looks something like $R_{nl}(r)Y_{lm}(\theta, \phi)$, and that all states with the same n and l quantum numbers have the same energy, *i.e.*, are $(2l + 1)$ -fold degenerate. The Hamiltonian for a free atom must have the symmetry of the full rotation group, as there are no special directions. Thus, we recall some properties of this group for the present discussion. First, we remark that the set of functions $\{Y_{lm} : m = -l, -l + 1, \dots, l - 1, l\}$ for a given l forms the basis for a $(2l + 1)$ -dimensional subspace which is invariant under the operations of the full rotation group. [A set $\{\psi_i\}$ of vectors is said to span an *invariant subspace* V_s under a given set of operations $\{P_j\}$ if $P_j\psi_i \in V_s \forall i, j$.] Furthermore, this subspace is “irreducible,” that is, it cannot be split into smaller subspaces which are also invariant under the rotation group.

Let us denote the linear transformation operator corresponding to element R of the rotation group by the symbol \hat{P}_R , *i.e.*:

$$\hat{P}_R f(\vec{x}) = f(R^{-1}\vec{x})$$

The way to think about this equation is to regard the left side as giving a “rotated function,” which we evaluate at point \vec{x} . The right side tells us

that this is the same as the original function evaluated at the point $R^{-1}\vec{x}$, where R^{-1} is the inverse of the rotation matrix corresponding to rotation R . Since $\{Y_{lm}\}$ forms an invariant subspace, we must have:

$$\hat{P}_R Y_{lm} = \sum_{m'=-l}^l Y_{lm'} D^l(R)_{m'm}$$

The expansion coefficients, $D^l(R)_{m'm}$, can be regarded as the elements of a matrix $D^l(R)$. As we have discussed in general, and as you may see more explicitly in the note on rotations in QM, D^l corresponds to an irreducible representation of the rotation group.

- (a) Prove, or at least make plausible, the fact that D^l is an irreducible representation of the rotation group. (Hint: You might show first that it is a representation and then show irreducibility by finding a contradiction with the supposition of reducibility).

Thus, for a free atom, we have that the degenerate eigenfunctions of a given energy must transform according to an irreducible representation of this group. If the eigenfunctions transform according to the l^{th} representation, the degeneracy of the energy level is $(2l + 1)$ (assuming no additional, “accidental” degeneracy).

- (b) We will need the character table of this group. Since all elements in the same class have the same character, we pick a convenient element in each class by considering rotations about the z -axis, $R = (\alpha, z)$ (means rotate by angle α about the z -axis). Thus:

$$\hat{P}_{(\alpha,z)} Y_{lm} = e^{-im\alpha} Y_{lm}$$

(which you should convince yourself of).

Find the character “table” of the rotation group, that is, find $\chi^\ell(\alpha)$, the character of representation D^ℓ for the class of rotations through angle α . If you find an expression for the character in the form of a sum, do the sum, expressing your answer in as simple a form as you can. Note that the answer is given in the text, just fill in the missing steps to your satisfaction.

- (c) Let us put our atom into a potential with cubic symmetry. Now the symmetry of the free Hamiltonian is broken, and we are left with the discrete symmetry of the cube. The symmetry group of proper rotations of the cube is a group of order 24 with 5 classes. Call this group “ O ”.

Construct the character table for O .

- (d) Consider in particular how the f -level ($l = 3$) of the free atom may split when it is placed in the “cubic potential”. The seven eigenfunctions which transform according to the irreducible representation D^3

of the full group will most likely not transform according to an irreducible representation of O . On the other hand, since the operations of O are certainly operations of D^3 , the eigenfunctions will generate some representation of O .

Determine the coefficients in the decomposition.

$$D^3 = a_1 O^1 \oplus a_2 O^2 \oplus a_3 O^3 \oplus a_4 O^4 \oplus a_5 O^5,$$

where O^i are the irreducible representations of O . Hence, show how the degeneracy of the 7-fold level may be reduced by the cubic potential. Give the degeneracies of the final levels.

Note that we cannot say anything here about the magnitude of any splittings (which could “accidentally” turn out to be zero!), or even about the ordering of the resulting levels – that depends on the details of the potential, not just its symmetry.