Chapter 3

Representation Theory

Solutions to Problems

3.1 Exercises

1. For the Poincare group $\bar{L}$, show that any element $\Lambda(M, z)$ can be written as a product of a pure homogeneous transformation followed by a pure translation. Also show that it can be written as a pure translation followed by a pure homogeneous transformation.

2. Show that the object $\{x, y\}$ defined in Eqn. **??** is a scalar product.

3. Carry out the steps to demonstrate the decomposition of a representation into irreps,
$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus \cdots \oplus a_{n_r} D^{(n_r)}, \tag{3.1}$$
with coefficients:
$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} \chi^{(j)*}(C_k)\chi(C_k)N_k. \tag{3.2}$$

4. Derive the constant in Eqn. **??**, that is, determine λ in:
$$\sum_{g \in C_k} D^{(i)}(g) = \lambda I. \tag{3.3}$$

   Solution: We can take the trace of both sides of this equation:
$$\mathrm{Tr}\left[\sum_{g \in C_k} D^{(i)}(g)\right] = \mathrm{Tr}\left[\lambda I\right] \tag{3.4}$$
$$\sum_{g \in C_k} \chi^{(i)}(g) = \lambda \ell_i \tag{3.5}$$

$$N_k\chi^{(i)}(C_k) = \lambda\ell_i, \tag{3.6}$$

or

$$\lambda = \frac{N_k}{\ell_i}\chi^{(i)}(C_k). \tag{3.7}$$

This is the result asserted in Eqn. **??**.

5. Show that the two irreps generated according to Figs. **??** and **??** are equivalent.

6. Consider the group of all rotations in two dimensions: $G = \{R(\theta) : 0 \le \theta < 2\pi\}$. As a linear operator on vectors in a two-dimensional Euclidean space, the elements of G may be represented (faithfully, or isomporphically) by the set of 2×2 matrices of the form:

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.8}$$

Show that this group can be decomposed into two one-dimensional representations, i.e., that you can find a transformation such that every element of G can be represented in the form:

$$D(\theta) = \begin{pmatrix} f(\theta) & 0 \\ 0 & g(\theta) \end{pmatrix}, \tag{3.9}$$

where the new representation is still faithful. You should find explicit expressions for f and g.

Hint: You want to find a similarity transformation, which just corresponds to a change in basis. You might consider the basis transformation so often encountered in quantum mechanics and in optics, corresponding to describing states in terms of "circular polarization" instead of "linear polarization".

7. Consider the dihedral group D_3, which is isomorphic with the group of permutations of three objects, S_3. Let V_2 be a two-dimensional Euclidean space spanned by orthonormal vectors $\mathbf{e}_x, \mathbf{e}_y$. Give the representation, D, of the elements of D_3 with respect to this basis. That is, express the transformed vectors $D(g)\mathbf{e}_i$ in terms of the original basis, and hence obtain representation D.

8. Consider the symmetry group, C_{4v}, of the square, consisting of rotations about the axis perpendicular to the square, and reflections about the vertical, horizontal, and diagonal axes in the plane of the square (but no mirror plane in the plane of the square).

   (a) Construct a suitable set of irreducible representations of C_{4v}. That is, up to equivalence, construct all of the irreducible representations of this group.

   Solution: The order of C_{4v} is $h = 8$, with $n_c = 5$ classes and elements:

i. $C_1 = \{e\}$, the identity.
ii. $C_2 = \{M_x, M_y\}$, mirror planes perpendicular to the square, containing the horizontal (x) and vertical (y) symmetry axes of the square, respectively.
iii. $C_3 = \{M_u, M_v\}$, mirror planes perpendicular to the square, containing the lines $x = y$ and $x = -y$, respectively.
iv. $C_4 = \{R_+, R_-\}$, rotations by ± 90 degrees about the principal axis of the square.
v. $C_5 = \{R\}$, rotation by 180 degrees about the principal axis of the square.

There are $n_r = n_c = 5$ irreps. We must have the sum of the squares of the dimensions equal to 8. Given that we know that the identity rep is of dimension 1, the possibilities are either $\{\ell_i\} = \{1,1,1,1,1,1,1,1\}$ or $\{1,1,1,1,4\}$. We may apply the reasoning we used in the case of the square drumhead in class to obtain:

i. $D^{(1)}(g) = 1,\ \forall g \in C_{4v}$.
ii. $D^{(2)}(\{e, R, R_\pm\}) = 1, D^{(}2)(\{M_x, M_y, M_u, M_v\}) = -1$.
iii. $D^{(3)}(\{e, R, M_x, M_y\}) = 1, D^{(}3)(\{R_\pm, M_u, M_v\}) = -1$.
iv. $D^{(4)}(\{e, R, M_u, M_v\}) = 1, D^{(}4)(\{R_\pm, M_x, M_y\}) = -1$.
v. $D^{(5)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D^{(5)}(M_x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$
$D^{(5)}(M_y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ D^{(5)}(M_u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$
$D^{(5)}(M_v) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ D^{(5)}(R) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$
$D^{(5)}(R_\pm) = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}.$

(b) Give the character table for C_{4v}.

Solution: Just take the traces of the irreps in part (a) to get the character table. The first four columns are just the irreps themselves.

9. In problem 6 you consider the reducibility of a two-dimensional representation of the group of rotations in two dimensions. We may remark that this is an abelian group. Let us generalize that result: Consider a group, G, with a unitary representation D, consisting of unitary matrices $D(g),\ g \in G$. If G is an abelian group, show that any such representation is, by a similarity transformation, equivalent to a representation by diagonal matrices (i.e., by a direct sum of 1×1 matrices). Note that we have already used group theory (Schur's lemma) to argue the truth of this. In this problem, I want you to use what you know about matrix theory to demonstrate the result.

10. Construct the character table for the tetrahedral symmetry group T_d. You may wish to keep a copy of your result for problem 12.

11. Let's take a peek at the relation of irreductible representations and the invariant subspaces of a vector space: Let V be the 6-dimensional function space consisting of polynomials of degree 2 in the two real variables x and y:
$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + h, \tag{3.10}$$
where a, b, c, d, e, h are complex numbers. If x, y transforms under the dihedral group D_3 (problem 7) as the coordinates of a 2-vector, then we obtain a 6-dimensional representation of D_3 on V. Identify the invariant subspaces of V under D_3, and the corresponding irreducible representations contained in this six dimensional representation (don't be afraid to use your intuition to make sure that what you find is sensible).

12. At last we are ready for a real physics application of group theory. We looked at the example of masses joined by springs in the shape of an equilateral triangle in this note. Now, let us consider the problem of four masses joined by springs. The four masses are at the corners of a tetrahedron, and the springs form the edges of the tetrahedron. Thus, there are six springs connecting the four masses. All four masses are equal, and all six springs are identical.

    We wish to determine the frequencies of the normal modes for this system. Notice that to solve the secular equation, $|V - \lambda I| = 0$, presents a formidable image. A little physical intuition can reduce it somewhat, but it would take real cleverness to solve it completely. This cleverness comes in the form of group theory! Group theory permits one to incorporate in a systematic and deliberate way everything we know about the symmetry of the problem, hence reducing it to a simpler problem.

    The problem is still not trivial − you should spend time thinking about convenient approaches in setting things up, and about ways to avoid doing unnecessary work. Above all, be careful, and check your results as you proceed. You already obtained the character table for the tetrahedral symmetry group in problem 10. This problem takes you the rest of the way through solving for the frequencies of the normal modes.

    (a) First step: Set up a 12-dimensional vector (coordinate system) describing the system, and derive the equations of motion, arriving finally at a set of linear equations that could be solved, in principle, to yield the frequencies of the normal modes.

    (b) Second step: Obtain the character table for the twelve-dimensional representation of the tetrahedral symmetry group that acts on your 12-dimensional vector describing the system. Decompose this representation into irreducible representations.

    (c) Final step: Obtain a small number of trace equations which you can use to solve to obtain the frequencies of the normal modes. Give the frequencies of the normal modes, and their degeneracies. Do your answers make physical sense?

13. The "quaternion" group consists of eight elements,

$$Q = \{1, -1, i, -i, j, -j, k, -k\}, \tag{3.11}$$

with multiplication table defined by (q is any element of Q):

$$\begin{aligned} 1q &= q \\ (-1)^2 &= 1 \\ (-1)q = q(-1) &= -q \\ i^2 = j^2 = k^2 = ijk &= -1 \end{aligned} \tag{3.12}$$

Find the character table for this group. Compare this character table with the character table for dihedral group D_4. Are these two groups isomorphic?

14. As a follow-on to the drumhead example in this note, consider the symmetry group of the regular pentagon, as given by a five-fold axis and several mirror planes. Do not include the mirror plan containing the plane of the pentagon itself (although you may amuse yourself by considering what happens if you add this operation).

    (a) List the group elements. Denote rotations with R's, and mirror operations with M's. Draw a picture! List the classes.

    (b) Construct the character table for the irreducible representations of this group.

    (c) Consider the mode of oscillation of a pentagonal drumhead where a nodal line extends from a vertex to the midpoint of the opposite side. Define (with pictures) a basis for the space generated by this mode and its degenerate partners. Give an explicit matrix for one element of each class of the group for the representation of the pentagonal symmetry group that is generated by these degenerate modes.

    (d) Decompose the representation found in part (c) into irreducible representations.

15. We would like to consider the (qualitative) effects on the energy levels of an atom which is moved from freedom to an external potential (a crystal, say) with cubic symmetry. Let us consider a one-electron atom and ignore spin for simplicity. Recall that the wave function for the case of the free atom looks something like $R_{nl}(r)Y_{lm}(\theta, \phi)$, and that all states with the same n and l quantum numbers have the same energy, *i.e.*, are $(2l + 1)$-fold degenerate. The Hamiltonian for a free atom must have the symmetry of the full rotation group, as there are no special directions. Thus, we recall some properties of this group for the present discussion. First, we remark that the set of functions $\{Y_{lm} \ : \ m = -l, -l+1, \cdots, l-1, l\}$ for a given l forms the basis for a $(2l + l)$-dimensional subspace which is invariant under the operations of the full rotation group. [A set $\{\psi_i\}$ of vectors is

said to span an *invariant subspace* V_s under a given set of operations $\{P_j\}$ if $P_j\psi_i \in V_s \ \forall i, j$.] Furthermore, this subspace is "irreducible," that is, it cannot be split into smaller subspaces which are also invariant under the rotation group.

Let us denote the linear transformation operator corresponding to element R of the rotation group by the symbol $\hat{P}_R$, *i.e.*:

$$\hat{P}_R f(\vec{x}) = f(R^{-1}\vec{x})$$

The way to think about this equation is to regard the left side as giving a "rotated function," which we evaluate at point $\vec{x}$. The right side tells us that this is the same as the original function evaluated at the point $R^{-1}\vec{x}$, where R^{-1} is the inverse of the rotation matrix corresponding to rotation R. Since $\{Y_{lm}\}$ forms an invariant subspace, we must have:

$$\hat{P}_R Y_{lm} = \sum_{m'=-1}^{l} Y_{lm'} D^l(R)_{m'm}$$

The expansion coefficients, $D^l(R)_{m'm}$, can be regarded as the elements of a matrix $D^l(R)$. As we have discussed in general, and as you may see more explicitly in the note on rotations in QM, D^ℓ corresponds to an irreducible representation of the rotation group.

(a) Prove, or at least make plausible, the fact that D^l is an irreducible representation of the rotation group. (Hint: You might show first that it is a representation and then show irreducibility by finding a contradiction with the supposition of reducibility).

Thus, for a free atom, we have that the degenerate eigenfunctions of a given energy must transform according to an irreducible representation of this group. If the eigenfunctions transform according to the l^{th} representation, the degeneracy of the energy level is $(2l + 1)$ (assuming no additional, "accidental" degeneracy).

(b) We will need the character table of this group. Since all elements in the same class have the same character, we pick a convenient element in each class by considering rotations about the z-axis, $R = (\alpha, z)$ (means rotate by angle α about the z-axis). Thus:

$$\hat{P}_{(\alpha,z)} Y_{\ell m} = e^{-im\alpha} Y_{\ell m}$$

(which you should convince yourself of).

Find the character "table" of the rotation group, that is, find $\chi^\ell(\alpha)$, the character of representation D^ℓ for the class of rotations through angle α. If you find an expression for the character in the form of a sum, do the sum, expressing your answer in as simple a form as you can. Note that the answer is given in the text, just fill in the missing steps to your satisfaction.

(c) Let us put our atom into a potential with cubic symmetry. Now the symmetry of the free Hamiltonian is broken, and we are left with the discrete symmetry of the cube. The symmetry group of proper rotations of the cube is a group of order 24 with 5 classes. Call this group "O".

Construct the character table for O.

(d) Consider in particular how the f-level ($l = 3$) of the free atom may split when it is placed in the "cubic potential". The seven eigenfunctions which transform according to the irreducible representation D^3 of the full group will most likely not transform according to an irreducible representation of O. On the other hand, since the operations of O are certainly operations of D^3, the eigenfunctions will generate some representation of O.

Determine the coefficients in the decomposition.

$$D^3 = a_1 O^1 \oplus a_2 O^2 \oplus a_3 O^3 \oplus a_4 O^4 \oplus a_5 O^5,$$

where O^i are the irreducible representations of O. Hence, show how the degeneracy of the 7-fold level may be reduced by the cubic potential. Give the degeneracies of the final levels.

Note that we cannot say anything here about the magnitude of any splittings (which could "accidentally" turn out to be zero!), or even about the ordering of the resulting levels – that depends on the details of the potential, not just its symmetry.