Chapter 5

The Permutation Group and Young Diagrams

5.1 Definitions

The permutation, or symmetric, group, S_n is interesting at least partly because it contains subgroups isomorphic to all groups of order $\leq n$. This result is known as "Cayley's theorem". It is also of great value in tensor analysis as the means to describe the tensor space in terms of symmetries under permutations of indicies. Here, we develop a diagrammatic approach to determining the irreducible representations of S_n, which will turn out to have applications beyond this immediate one.

Recall that we can express an element of S_n in cycle notation. For example, the element of S_5:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} \tag{5.1}$$

is described in cycle notation as (142)(35). We can make a useful correspondence between cycle structures and the "partitions" of integer n:

Def: A *partition* of a positive integer n is a set of integers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0, \tag{5.2}$$

and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = n. \tag{5.3}$$

Consider the class structure of the symmetric group S_n. Classes are given by cycle structures, *i.e.*, a particular class is specified by giving the n numbers $\omega_1, \omega_2, \ldots, \omega_n$, where ω_i is the number of i cycles in an element belonging to the class. Thus, for $(142)(35) \in S_5$, $\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = 1$, $\omega_4 = \omega_5 = 0$. Noticing that $\sum_{i=1}^{n} i\omega_i = n$, we see that the specification of a class of the symmetric group corresponds to the specification of a partition of n, according to the construction:

$$\begin{aligned}
\lambda_1 &= \omega_1 + \omega_2 + \ldots \omega_n \\
\lambda_2 &= \qquad \omega_2 + \ldots \omega_n \\
&\vdots \\
\lambda_n &= \qquad\qquad \omega_n.
\end{aligned}$$

For our S_5 example:

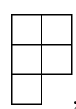
$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 2 \\
\lambda_3 &= 1 \\
\lambda_4 &= 0 \\
\lambda_5 &= 0,
\end{aligned} \tag{5.4}$$

and the sum of these numbers is five.

We use this correspondence in the invention of a graphical description known as Young Diagrams.

Def: A *Young Diagram* is a diagram with n boxes arranged in n rows corresponding to a partition of n, *i.e.*, with row i containing λ_i boxes.

For example, the diagram:



, (5.5)

for S_5 corresponds to $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 1$, and $\lambda_4 = \lambda_5 = 0$. Because of the ordering of the λ's, each row of a Young diagram has at most as many boxes as the row above it.

Note that giving all the Young diagrams for a given n classifies all of the classes of S_n. Since $n_c = n_r$ it may not be surprising that such diagrams are also useful in identifying irreducible representations of S_n. That is, there is a $1:1$ correspondence between Young diagrams and irreducible representations of S_n. Furthermore, these diagrams are useful in decomposing products of irreducible representations.

Def: A *Young tableau* is a Young diagram in which the n boxes have been filled with the numbers $1, \ldots, n$, each number used exactly once.

For example:

$$\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad . \tag{5.6}$$

There are $n!$ Young tableau for a given Young diagram.

Def: A *standard Young tableau* is a Young tableau in which the numbers appear in ascending order within each row or column from left to right and top to bottom.

For example, the following are the possible standard Young tableau with the given shape:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;, \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \; . \tag{5.7}$$

Def: A *normal tableau* is a standard Young tableau in which the numbers are in order, left to right and top to bottom.

There is only one normal tableau of a given shape, e.g.,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad . \tag{5.8}$$

From a normal tableau, we may obtain all other standard tableau by suitable permutations, for example:

$$Y_1 : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xrightarrow{(1)(2453)} Y_2 : \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} . \tag{5.9}$$

That is, $(2453)Y_1 = Y_2$.

5.2 Examples: System of particles

Suppose we have a system of n identical particles, in which the Hamiltonian, H, is invariant under permutations of the particles. Let x_i be the coordinate (position, spin, etc.) of particle i. Suppose $\psi(x_1, x_2, \ldots, x_n)$ is an eigenfunction of H belonging to eigenvalue E. Then any permutation of the particles:

$$\begin{aligned} P_a\psi &= \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \psi(x_1, x_2, \ldots, x_n) \\ &= \psi(x_{a_1}, x_{a_2}, \ldots, x_{a_n}) \end{aligned} \tag{5.10}$$

is another eigenfunction belonging to the same eigenvalue.

In quantum mechanics, we have symmetric wave functions, under interchange of any pair of particle coordinates, for bosons, and anti-symmetric wave functions for fermions. Define a "symmetrizer operator" by:

$$S \equiv \frac{1}{n!} \sum_P P, \tag{5.11}$$

where $\sum_P$ is short for $\sum_{P \in S_n}$, that is a sum over all permutations of the n particle coordinates. Likewise, define an "anti-symmetrizer operator" by:

$$A \equiv \frac{1}{n!} \sum_P \delta_P P, \tag{5.12}$$

where

$$\delta_P \equiv \begin{cases} +1 & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd.} \end{cases} \tag{5.13}$$

We call δ_P the "parity" of the permutation. It is given by

$$\delta_P = (-1)^q, \tag{5.14}$$

where q is the number of transpositions required to produce permutation P starting from the normal tableau.

It is an exercise for the reader to show that a k-cycle has parity $(-1)^{k-1}$. Therefore, if a permutation, P, consists of ℓ cycles with structure $\{k_1, k_2, \ldots, k_\ell\}$ then the parity of P is:

$$\begin{aligned} \delta_P &= (-1)^{\sum_{i=1}^{\ell}(k_i-1)} \\ &= (-1)^{n-\ell}, \end{aligned} \tag{5.15}$$

where the second line follows because $\sum_{i=1}^{\ell} k_i = n$. The quantity $n - \ell$ is called the *decrement* of P.

We also leave it as an exercise for the reader to show that, for any $P_a \in S_n$:

$$P_a S = S \tag{5.16}$$

$$P_a A = A P_a = \delta_{P_a} A \tag{5.17}$$

$$S^2 = S \tag{5.18}$$

$$A^2 = A. \tag{5.19}$$

Thus, S and A act as projection operators.

Consider two-particle states. Let u and d be orthogonal single-particle states[1], and $\psi_N = u(x_1)d(x_2)$. We have symmetrizer:

$$S_{12} = \frac{1}{2}(e + P_{12}), \tag{5.20}$$

[1] Alternatively, we could be talking about the two angular momentum states of a spin-1/2 system, with u corresponding, say to spin "up", and d to spin "down".

where e is the identity operator of S_2. The operator S_{12} projects out the symmetric part of ψ_N:

$$\psi^S \equiv S_{12}\psi_N = \frac{1}{2}\left[u(x_1)d(x_2) + d(x_1)u(x_2)\right]. \tag{5.21}$$

Likewise, the anti-symmetrizer,

$$A_{12} = \frac{1}{2}(e - P_{12}), \tag{5.22}$$

projects out the antisymmetric piece:

$$\psi^A \equiv A_{12}\psi_N = \frac{1}{2}\left[u(x_1)d(x_2) - d(x_1)u(x_2)\right]. \tag{5.23}$$

Note that the combinations $u(x_1)u(x_2)$ and $d(x_1)d(x_2)$ are already symmetric.

Now we relate this discussion to our graphical formalism. The Young diagram $\square\square$ corresponds to the class of two 1-cycles, that is, the identity of S_2. The Young diagram $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ corresponds to the class of one 2-cycle, that is transposition. Thus, we make the identification of tableau:

$$\boxed{1}\boxed{2} \text{ with } \begin{cases} u(x_1)u(x_2) \\ \frac{1}{2}\left[u(x_1)d(x_2) + d(x_1)u(x_2)\right] \\ d(x_1)d(x_2), \end{cases} \tag{5.24}$$

and

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \text{ with } \frac{1}{2}\left[u(x_1)d(x_2) - d(x_1)u(x_2)\right]. \tag{5.25}$$

That is, two boxes in a row correspond to a symmetric state, and two in a column to an antisymmetric state.

Let's try this with three-particle states, with u, d, s as orthonormal single particle states. We'll drop the x from our notation, and simply write $\psi_N = u(1)d(2)s(3)$. There are $3! = 6$ linearly independent functions obtained by permuting the $1, 2, 3$ particle labels, or by permuting the state labels u, d, s. We'll do the latter, and also simplify our notation still further and drop the particle labels, with the understanding that they remain in the order 123.

We rewrite the six linearly independent functions obtained by permutations into a different set of six linearly independent functions, based on symmetry properties under interchange. First, the completely symmetric arrangement:

$$\begin{aligned} \psi^S &= S_{123}\psi_N = S_{123}uds \\ &= \frac{1}{3!}(e + P_{12} + P_{13} + P_{23} + P_{123} + P_{132})uds \\ &= \frac{1}{3!}(uds + dus + sdu + usd + dsu + sud). \end{aligned} \tag{5.26}$$

Once again, this corresponds to the identity class of three 1-cycles: $\boxed{1}\boxed{2}\boxed{3}$. This symmetric state is invariant under the actions of S_3, hence it generates the one-dimensional identity representation.

The completely antisymmetric arrangement is:

$$\begin{aligned}
\psi^A &= A_{123}\psi_N = A_{123}uds \\
&= \frac{1}{3!}(e - P_{12} - P_{13} - P_{23} + P_{123} + P_{132})uds \\
&= \frac{1}{3!}(uds - dus - sdu - usd + dsu + sud), \qquad (5.27)
\end{aligned}$$

corresponding to $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$. All actions of the S_3 group on ψ^A yield ± 1 times ψ^A. Thus, this function is a vector in another one-dimensional invariant subspace under the actions of the group elements and hence generates another one-dimensional irreducible representation of S_3. Note that it is not equivalent to the identity representation.

There are four more functions to build; these must have mixed symmetry. We may proceed by symmetrizing *uds* with respect to two particles, and then antisymmetrizing with respect to two particles (or vice versa), with one particle in common between the two operations.[2] There is some arbitrariness in how we choose to carry out this program. Let us take:

$$\begin{aligned}
\psi_1 &= A_{13}S_{12}\psi_N = A_{13}\frac{1}{2}(uds + dus) = \frac{1}{4}(uds - sdu + dus - sud) \\
\psi_2 &= A_{23}S_{12}\psi_N = \frac{1}{4}(uds - usd + dus - dsu) \\
\psi_3 &= S_{13}A_{12}\psi_N = \frac{1}{2}(e + P_{13})\frac{1}{2}(uds - dus) = \frac{1}{4}(uds + sdu - dus - sud) \\
\psi_4 &= S_{23}A_{12}\psi_N = \frac{1}{4}(uds + usd - dus - dsu). \qquad (5.29)
\end{aligned}$$

We note that ψ_1 and ψ_2 form an invariant subspace under S_3:

$$\begin{aligned}
(12)\psi_1 &= \frac{1}{4}(dus - dsu + uds - usd) = \psi_2 \\
(13)\psi_1 &= -\psi_1 \qquad (5.30) \\
(12)\psi_2 &= \psi_1 \\
(13)\psi_2 &= \frac{1}{4}(sdu - dsu + sud - usd) = \psi_2 - \psi_1,
\end{aligned}$$

with the other S_3 elements obtained by products of these. Likewise, ψ_3 and ψ_4 form an invariant subspace.

Typically, we want to form an orthogonal system. We may check whether our states are orthogonal. For example,

$$\begin{aligned}
(\psi_1, \psi_3) &= (A_{13}S_{12}uds, S_{13}A_{12}uds) \\
&= (uds, S_{12}A_{13}S_{13}A_{12}uds) = 0, \qquad (5.31)
\end{aligned}$$

[2]Note that

$$S_{ij}A_{ij} = \frac{1}{2}(e + P_{ij})\frac{1}{2}(e - P_{ij}) = \frac{1}{4}(e + P_{ij} - P_{ij} - e) = 0. \qquad (5.28)$$

That is, our projections project onto orthogonal subspaces.

since $A_{13}S_{13} = 0$. Likewise, we find that

$$(\psi_1, \psi_4) = (\psi_2, \psi_3) = (\psi_2, \psi_4) = 0. \tag{5.32}$$

However, we also find that $(\psi_1, \psi_2) \neq 0$ and $(\psi_3, \psi_4) \neq 0$, so our $\psi_1, \psi_2, \psi_3, \psi_4$ states do not yet form an orthogonal system. But we make take linear combinations (a, b, c, d are normalization constants):

$$\begin{aligned}
\psi_1' &= a(\psi_1 + \psi_2) \\
&= -\frac{1}{\sqrt{12}}(2uds + 2dus - sdu - sud - usd - dsu), \qquad (5.33)\\
\psi_2' &= b(\psi_1 - \psi_2) \\
&= -\frac{1}{2}(usd + dsu - sdu - sud), \qquad (5.34)\\
\psi_3' &= c(\psi_3 + \psi_4) \\
&= \frac{1}{\sqrt{12}}(2uds - 2dus + sdu - sud + usd - dsu), \qquad (5.35)\\
\psi_4' &= d(\psi_3 - \psi_4) \\
&= \frac{1}{2}(-sdu + sud + usd - dsu), \qquad (5.36)
\end{aligned}$$

where we have normalized and adopted phase conventions.

Thus, we have a set of six orthonormal functions. Both ψ_1' and ψ_4' are symmetric under the transposition (12), hence both correspond to the Young tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. Likewise, ψ_2' and ψ_3' are antisymmetric under (12), corresponding to tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. The states $\{\psi_1', \psi_2'\}$ form an invariant subspace under S_3, and the states $\{\psi_3', \psi_4'\}$ form another invariant subspace. Both subspaces lead to the same irreducible representation of S_3, a "mixed" representation (that is, neither purely symmetric nor purely antisymmetric under transpositions), with Young diagram $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$. This is a two-dimensional representation, acting on either of the two-dimensional invariant subspaces. For example,

$$\begin{aligned}
(12)\psi_1' &= \psi_1' \\
(12)\psi_2' &= -\psi_2'
\end{aligned} \tag{5.37}$$

tells us that the (12) element is represented in this basis by:

$$D(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.38}$$

Now notice that we are also generating an orthonormal basis of states of the $3 \otimes 3 \otimes 3$ representation of $SU(3)$! We thus have a connection between $SU(3)$ and the permutation symmetry. Let us pursue this idea further in this example. We'll make the example more concrete by interpreting that the particles u, d, s

as quark flavor eigenstates. In this case, we are generating the wavefunctions of the baryons, in terms of quark flavor content. Now $3 \times 3 \times 3 = 27$, so we are dealing with a 27-dimensional representation of $SU(3)$. We proceed to find the decomposition of this into irreducible representations, and obtain the baryon flavor wavefunctions:

First, we have,

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \psi^A = \frac{1}{6}(uds - dus + sud - usd + dsu - sdu) = \begin{array}{|c|}\hline u \\ \hline d \\ \hline s \\ \hline \end{array}, \tag{5.39}$$

where the graph on the left is our familiar Young tableau indicating complete antisymmetry under transpositon of coordinates in S_3. The graph on the right, called a "Weyl tableau", indicates that the wave function is also completely antisymmetric under interchange of *flavors* in $SU(3)$. This is the only completely antisymmetric state: Any "rotation" in $SU(3)$ gives back this state. Hence this generates a one-dimensional representation of $SU(3)$, the identity representation.

We have seen that we also have states with mixed symmetry under the actions of S_3. We found four such states comprised of uds, two associated with $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ and two with $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$. There are, in addition states with two identical quarks. For example, we may obtain the $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ uud state by letting $s \to u$ in ψ'_1:

$$\begin{aligned} \psi'_1 &= -\frac{1}{\sqrt{12}}(2uds + 2dus - sdu - sud - usd - dsu), \\ &\to -\frac{1}{\sqrt{12}}(2udu + 2duu - udu - uud - uud - duu), \\ &= -\frac{1}{\sqrt{12}}(udu + duu - 2uud), \\ &\to -\frac{1}{\sqrt{6}}(udu + duu - 2uud), \end{aligned} \tag{5.40}$$

where we have normalized to one in the last step.

Similarly, the $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ uud state is obtained from ψ'_2:

$$\begin{aligned} \psi'_2 &= -\frac{1}{2}(usd + dsu - sdu - sud), \\ &\to -\frac{1}{2}(uud + duu - udu - uud), \\ &\to -\frac{1}{\sqrt{2}}(udu - duu). \end{aligned} \tag{5.41}$$

Notice that we get the same state by replacing the s quark in ψ'_3 with a u quark. Likewise, ψ'_4 gives the same state as ψ'_1.

Let us summarize the mixed symmetry states of the baryons:[3]

Baryon name	$\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$ (12) symmetric	$\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$ (12) antisymmetric	Weyl tableau
N^+	$-\frac{1}{\sqrt{6}}(udu + duu - 2uud)$	$\frac{1}{\sqrt{2}}(udu - duu)$	$\begin{smallmatrix} u & u \\ d & \end{smallmatrix}$
N^0	$\frac{1}{\sqrt{6}}(udd + dud - 2ddu)$	$\frac{1}{\sqrt{2}}(udd - dud)$	$\begin{smallmatrix} u & d \\ d & \end{smallmatrix}$
Σ^+	$\frac{1}{\sqrt{6}}(usu + suu - 2uus)$	$\frac{1}{\sqrt{2}}(usu - suu)$	$\begin{smallmatrix} u & u \\ s & \end{smallmatrix}$
Σ^0	$-\frac{1}{\sqrt{12}}(2uds + 2dus - sdu - sud - usd - dsu)$	$-\frac{1}{2}(usd + dsu - sdu - sud)$	$\begin{smallmatrix} u & d \\ s & \end{smallmatrix}$
Σ^-	$\frac{1}{\sqrt{6}}(sdd + dsd - 2dds)$	$\frac{1}{\sqrt{2}}(sdd - dsd)$	$\begin{smallmatrix} d & d \\ s & \end{smallmatrix}$
Λ^0	$\frac{1}{2}(sud - sdu + usd - dsu)$	$\frac{1}{\sqrt{12}}(2uds - 2dus + sdu - sud + usd - dsu)$	$\begin{smallmatrix} u & s \\ d & \end{smallmatrix}$
Ξ^0	$-\frac{1}{\sqrt{6}}(uss + sus - 2ssu)$	$-\frac{1}{\sqrt{2}}(uss - sus)$	$\begin{smallmatrix} u & s \\ s & \end{smallmatrix}$
Ξ^-	$-\frac{1}{\sqrt{6}}(dss + sds - 2ssd)$	$-\frac{1}{\sqrt{2}}(dss - sds)$	$\begin{smallmatrix} d & s \\ s & \end{smallmatrix}$

Thus, we have two eight-dimensional irreducible representations of $SU(3)$ with mixed symmetry. Together with the completely antisymmetric state, we so far have irreducible representations of dimensions $1, 8, 8$ in our 27-dimensional $3 \otimes 3 \otimes 3$ product representation. We next consider the representation generated by the completely symmetric states, corresponding to $\begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix}$. We may start with:

$$\psi^S = \frac{1}{\sqrt{6}}(uds + dus + sdu + usd + dsu + sud). \tag{5.42}$$

The particle name attached to this state is Σ^{*0}. If we replace the s by a u, for example, we get

$$\frac{1}{\sqrt{3}}(udu + uud + duu), \tag{5.43}$$

known as Δ^+.

We summarize the symmetric states in a table:

Baryon name	$\begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix}$ completely symmetric	Weyl tableau
Δ^{++}	uuu	$\begin{smallmatrix} u & u & u \end{smallmatrix}$

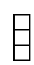
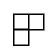
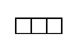
[3]Note that the superscripts give the electric charges of the states, where the u has charge $\frac{2}{3}$ and the d and s both have charge $-\frac{1}{3}$. Thus the charge operator, Q is related to the I_3 and Y operators by $Q = I_3 + \frac{Y}{2}$.

Δ^+	$\frac{1}{\sqrt{3}}(udu + uud + duu)$	$\boxed{u}\boxed{u}\boxed{d}$
Δ^0	$\frac{1}{\sqrt{3}}(udd + dud + ddu)$	$\boxed{u}\boxed{d}\boxed{d}$
Δ^-	ddd	$\boxed{d}\boxed{d}\boxed{d}$
Σ^{*+}	$\frac{1}{\sqrt{3}}(uus + usu + suu)$	$\boxed{u}\boxed{u}\boxed{s}$
Σ^{*0}	$\frac{1}{\sqrt{6}}(uds + dus + sdu + usd + dsu + sud)$	$\boxed{u}\boxed{d}\boxed{s}$
Σ^{*-}	$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$	$\boxed{d}\boxed{d}\boxed{s}$
Ξ^{*0}	$\frac{1}{\sqrt{3}}(uss + sus + ssu)$	$\boxed{u}\boxed{s}\boxed{s}$
Ξ^{*-}	$\frac{1}{\sqrt{3}}(dss + sds + ssd)$	$\boxed{d}\boxed{s}\boxed{s}$
Ω^-	sss	$\boxed{s}\boxed{s}\boxed{s}$

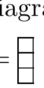
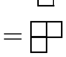
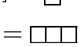
There are thus ten symmetric states, generating a ten-dimensional irreducible representation of $SU(3)$. We have once again found that $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ in $SU(3)$. Notice that we can generate all of the irreducible representations and bases from the "Weyl" diagrams, with two simple rules:

1. No column contains the same label twice.
2. Within each row or column, the state labels must be in non-decreasing order (according to whatever convention is chosen for the ordering of u, d, s).

Let us notice something now: When we formed the $3 \otimes 3 \otimes 3$ product representation of $SU(3)$, we obtained the Clebsch-Gordan series consisting of $SU(3)$ irreducible representations:

Number of irreps	Dimension of irrep	Young diagram
1	1	
2	8	
1	10	

But we also obtained irreducible representations of S_3. That is, we obtained the decomposition of our 27-dimensional representation of S_3, acting on our 27-dimensional state space, into the irreducible representaions of S_3:

Number of irreps	Dimension of irrep	Young diagram
1	1	$[1^3] =$ 
8	2	$[12] =$ 
10	1	$[3] =$ 

Here we have introduced the notation $[a^i b^j \ldots]$ to stand for a partition of n with i occurrences of "a", j occurrences of "b", etc. The first one-dimensional representation acts on the completely antisymmetric basis vector, the eight two-dimensional representations act on the vectors of mixed symmetry, and the final ten one-dimensional representations act on each of the ten symmetric basis vectors.

We notice a kind of "duality" between the number of irreducible representations of $SU(3)$ and the dimensions of the S_3 irreducible representations, and vice versa. This result holds more generally than this example. The general statement is:

Theorem: The multiplicity of the irreducible representation $[f]$ of S_n, denoted by $m_{[f]}(S_n)$ is equal to the dimension of the irreducible representation $[f]$ of $SU(N)$, denoted by $d_{[f]}\left(SU(N)\right)$:

$$m_{[f]}(S_n) = d_{[f]}\left(SU(N)\right), \tag{5.44}$$

and vice versa:

$$m_{[f]}\left(SU(N)\right) = d_{[f]}(S_n), \tag{5.45}$$

in the same tensor space of dimension N^n.

We have introduced the language of a "tensor space" here, we'll define and discuss this in the next section.

We conclude this section with an important theorem on the irreducible representations of S_n, generalizing the observations we have made for S_2 and S_3. We introduce the notation Θ_λ to refer to the normal Young tableau associated with partition of n specified by $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We let Θ_λ^p refer to the standard tableau obtained by permutation p on Θ_λ.

Now define:

Def: The *irreducible symmetrizer*, or *Young symmetrizer*, e_λ^p associated with the Young tableau Θ_λ^p is

$$e_\lambda^p \equiv \sum_{h,v} \delta_v h v, \tag{5.46}$$

where h is a horizontal permutaion of Θ_λ^p and v is a vertical permutaion.

An example should help to make this clear. Consider S_3. We have (up to a factor of 3! for S and A):

$$\Theta_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} : e_3 = \sum_h h = \sum_{p\in S_3} p = S \tag{5.47}$$

$$\Theta_{21} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} : e_{21} = \left[(e + (12)\right]\left[e - (13)\right]$$

$$= e + (12) - (13) - (132) \tag{5.48}$$

$$\Theta_{21}^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} : e_{21}^{(23)} = \left[(e + (13)\right]\left[e - (12)\right]$$

$$= e + (13) - (12) - (123) \tag{5.49}$$

$$\Theta_{1^3} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} : e_{1^3} = \sum_v \delta_v v = \sum_{p\in S_3} \delta_p p = A. \tag{5.50}$$

This exhausts the standard tableau for S_3.

We are ready for the theorem, which tells us that these irreducible symmetrizers generate the irreducible representations of S_n:

Theorem: The irreducible symmetrizers $\{e_\lambda\}$ associated with the normal Young tableau $\{\Theta_\lambda\}$ generate all of the inequivalent irreducible representations of S_n.

The general proof of this may be found in Tung and in Hamermesh. We'll make some observations here:

1. The number of inequivalent irreducible representations of S_n is given by the number of different Young diagrams, since they can be put into 1:1 correspondence with the classes.
2. There is one e_λ for each Young diagram, since there is one normal tableau for each diagram. Thus, the number of elements of $\{e_\lambda\}$ is the number of irreducible representations.
3. The remainder of the proof requires showing that each e_λ generates an inequivalent irreducible representation.

Notice that a corollary to this theorem is the fact that e_λ and e_λ^p generate equivalent irreducible representations. We may further notice that the dimension of an irreducible representation $[f]$ of S_n is equal to the number of standard Young tableaux associated with $[f] = [f_1 f_2 \ldots f_n]$. For example, in S_3, $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ generates a one-dimensional representation, $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ generate a two-dimensional representation, and $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ generates a one-dimensional representation. We may check that $1^2 + 2^2 + 1^2 = 6$, the order of S_3.

5.3 Tensors and tensor spaces

Def: Let V be an N-dimensional vector space:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \in V. \tag{5.51}$$

The product of n vectors: $x(1) \otimes x(2) \otimes \cdots \otimes x(n)$ forms a *tensor of rank n* in a *tensor space* of N^n dimensions. That is the direct product space; $V \otimes V \otimes \cdots \otimes V$ is called a tensor space.

We may denote the N^n tensor components by:

$$T_{i_1 i_2 \cdots i_n} = x_{i_1}(1) x_{i_2}(2) \cdots x_{i_n}(n), \tag{5.52}$$

where the indices $i_1, \ldots, i_n$ range over $1, 2, \ldots, N$.

Let G be a continuous group of linear transformations on V:

$$x \xrightarrow[a \in G]{} x' \;\Rightarrow\; x' = ax, \tag{5.53}$$

where $a \in G$ is an $N \times N$ matrix (depending on the parameters of group G). Under the action of $a \in G$, the tensor components transform according to:

$$T'_{i_1 i_2 \cdots i_n} = a_{i_1 i'_1} a_{i_2 i'_2} \cdots a_{i_n i'_n} T_{i'_1 i'_2 \cdots i'_n}, \tag{5.54}$$

where it is understood that repeated indices are summed over.

Notice the connection with direct product representations: In the tensor space, the transformation $a \in G$ is represented by $N^n \times N^n$ component matrix:

$$D(a) = a \otimes a \otimes a \cdots \otimes a, \tag{5.55}$$

with components

$$D(a)_{i_1 i_2 \cdots i_n, i'_1 i'_2 \cdots i'_n} = a_{i_1 i'_1} a_{i_2 i'_2} \cdots a_{i_n i'_n}. \tag{5.56}$$

This is a generalization of our earlier discussion on direct product matricies.

The representation $D(a)$ is generally reducible with respect to both G and S_n, the latter corresponding to symmetries with respect to permutations of the indicies. For a tensor of rank $n = 1$ the relevant symmetric group is S_1. Hence the components of a vector x which form a tensor of rank one correspond to the Young diagram $\square$.

Now consider the second rank tensor $T_{i_1 i_2}$. Permuting the indicies gives $T_{i_2 i_1}$. We may form:

$$T_{i_1 i_2} \pm T_{i_2 i_1}, \tag{5.57}$$

forming the basis of the symmetric and antisymmetric product representations, described by the Young diagrams $\square\square$ and $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$. The indicies i_1 and i_2 run from 1 to N. The matrix $D(a) = a \otimes a$ may be reduced to the direct sum of an antisymmetric representation and a symmetric representation. The antisymmetric representation (of S_2) has dimension

$$d_A = \frac{N(N-1)}{2}. \tag{5.58}$$

We may see this as follows: The index i_1 takes on values $1, \ldots, N$. For each i_1, i_2 can take on $N - 1$ values different from i_1. But each $T_{i_1 i_2} - T_{i_2 i_1}$ occurs twice (with opposite sign) in this counting, hence the factor of 1/2. This leaves a symmetric representation with dimension

$$d_S = N^2 - \frac{N(N-1)}{2} = \frac{N(N+1)}{2}. \tag{5.59}$$

Notice that the interchange of i_1 with i_2 corresponds to transposition $p = (12)$ on $T'_{i_1 i_2} = a_{i_1 i'_1} a_{i_2 i'_2} T_{i'_1 i'_2}$, and therefore:

$$\begin{aligned} pT'_{i_1 i_2} = T'_{i_2 i_1} &= a_{i_2 i'_2} a_{i_1 i'_1} T_{i'_2 i'_1} \\ &= a_{i_1 i'_1} a_{i_2 i'_2} T_{i'_2 i'_1} \\ &= a_{i_1 i'_1} a_{i_2 i'_2} pT_{i'_1 i'_2}. \end{aligned} \quad (5.60)$$

Thus, any $a \in G$ commutes with $p \in S_2$. This property remains valid for n^{th} rank tensors: Let $p \in S_n$, and

$$T_{i_1 i_2 \cdots i_n} = T_{(i)} = x_{i_1}(1) x_{i_2}(2) \cdots x_{i_n}(n), \quad (5.61)$$

where we have introduced a shorter notation for the indices. Then

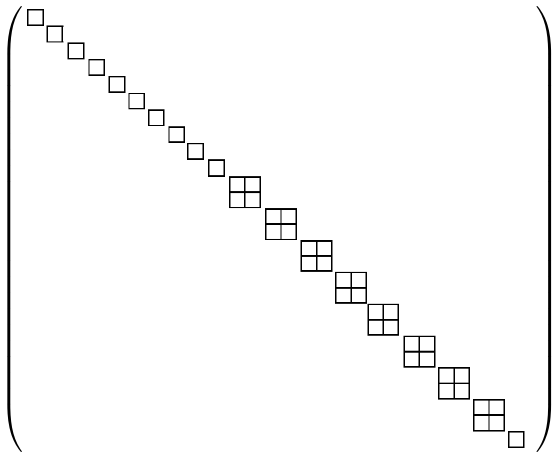
$$\begin{aligned} (pT)_{(i)} &= x_{i_1}(a_1) x_{i_2}(a_2) \cdots x_{i_n}(a_n) \\ &= T_{p(i)}, \end{aligned} \quad (5.62)$$

since the permutation of the n objects $1, 2, \ldots, n$ is equivalent to the permutation of the indicies $i_1, i_2, \ldots, i_n$. Now,

$$\begin{aligned} (pT')_{(i)} = T'_{p(i)} &= D_{p(i)p(j)} T_{p(j)} \\ &= D_{p(i)p(j)} \left(pT\right)_{(j)} \\ &= D_{(i)(j)} \left(pT\right)_{(j)}, \end{aligned} \quad (5.63)$$

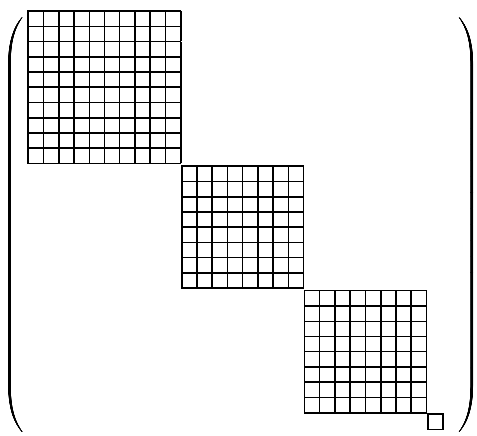
since $D(a)$ is bisymmetric, that is invariant under the simultaneous identical permutations of both the i's and j's.

Thus, any $p \in S_n$ commutes with any transformation of linear operator G on the tensor space. This is an important observation. It means that linear combinations which have a particular permutation symmetry transform among themselves, and can also be described by Young tableaux associated with the same Young diagram, generating an invariant subspace of S_n. The space of an n-rank tensor is reducible into subspaces of tensors of different symmetries. A tensor space can be reduced with respect to both G and S_n, and a kind of duality between a linear group G and a symmetric group S_n exists in a tensor space. We noted this earlier in our $3 \otimes 3 \otimes 3$ example under $SU(3)$. The 27-dimensional representation of S_3 has the reduction to block diagonal form:



. (5.64)

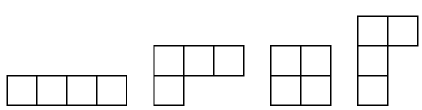
The boxes here indicate possibly non-zero components of the matrix, not Young diagrams! Likewise, the 27-dimensional representation of $SU(3)$ has the reduction to block diagonal form:



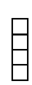
. (5.65)

For another example, consider the 3^4-dimensional tensor space, generated by (u, d, s) vectors in direct products of rank four (that is, $N = 3$, $n = 4$). Let us determine the multiplicities of the irreducible representations of $SU(3)$ and S_4 in the decompositions of the representations on this tensor space.

The irreducible representations accepted in this space by both groups have Young diagrams:



4 31 2^2 211

Notice that the totally antisymmetric representation  does not appear, because we cannot make a totally antisymmetric combination under S_4 from three distinct components (u, d, s).

Under S_4, the dimensions of the surviving irreducible representations are:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array} \qquad d_4^{S_4} = 1 \quad \Rightarrow \quad m_4^{SU(3)} = 1$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad d_{31}^{S_4} = 3 \quad \Rightarrow \quad m_{31}^{SU(3)} = 3$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad d_{22}^{S_4} = 2 \quad \Rightarrow \quad m_{22}^{SU(3)} = 2$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad d_{211}^{S_4} = 3 \quad \Rightarrow \quad m_{211}^{SU(3)} = 3$$

To determine the multiplicities under S_4, or the dimensions under $SU(3)$, we could do the same sort of constructive analysis as we did for $2 \otimes 2$ under $SU(2)$ or $3 \otimes 3 \otimes 3$ under $SU(3)$. For example, the dimension $d_{211}^{SU(3)}$ is clearly 3, since $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline\end{array}$ is completely antisymmetric in (u, d, s), hence of dimension one, and adding one more u, d, or s gets us to three dimensions. Likewise, for the diagram $\begin{array}{|c|c|c|c|}\hline \, & \, & \, & \, \\ \hline\end{array}$ we have a 15-dimensional representation of $SU(3)$, with a set of linearly independent vectors:

$uuuu$
$dddd$
$ssss$
$uuud + uudu + uduu + duuu$
$uuus + uusu + usuu + suuu$
$dddu + ddud + dudd + uddd$
$ddds + ddsd + dsdd + sddd$
$sssu + ssus + suss + usss$
$sssd + ssds + sdss + dsss$
$uudd + udud + uddu + duud + dudu + dduu$
$uuss + usus + ussu + suus + susu + ssuu$

$ddss + dsds + dssd + sdds + sdsd + ssdd$
$udsu + udus + uuds + usdu + usud + uusd + sudu + suud + dusu + duus + dsuu + sduu$
$udsd + udds + duds + usdd + dusd + sudd + sdud + dsud + ddus + dsdu + ddsu + sddu$
$udss + usds + suds + ussd + susd + ssud + duss + dsus + sdus + dssu + sdsu + ssdu$

We could also use the general formula:

$$d_{[f]}^{SU(N)} = \prod_{i<j}^{N} \frac{f_i - f_j + j - i}{j - i}. \tag{5.66}$$

For example, for $\square\square\square\square$, $f = (4, 0, 0, 0)$ and

$$d_{[4]}^{SU(3)} = \left(\frac{4+1}{1}\right)\left(\frac{4+2}{2}\right)\left(\frac{0+1}{1}\right) = 15, \tag{5.67}$$

remembering that there is no $j = 4$ contribution since $N = 3$. Likewise,

$$d_{[31]}^{SU(3)} = \left(\frac{3-1+1}{1}\right)\left(\frac{3+2}{2}\right)\left(\frac{1+1}{1}\right) = 15, \tag{5.68}$$

$$d_{[22]}^{SU(3)} = \left(\frac{2+1}{1}\right)\left(\frac{2+2}{2}\right) = 6, \tag{5.69}$$

$$d_{[211]}^{SU(3)} = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right) = 3. \tag{5.70}$$

Notice that

$$15 \times 1 + 15 \times 3 + 6 \times 2 + 3 \times 3 = 81 = 3^4, \tag{5.71}$$

so all dimensions in the representation are accounted for in our reduction to irreducible representations. We notice that there are no singlets in this decomposition. A physical application of this is in $SU(3)_{\text{color}}$, where we find that no colorless (i.e., color singlet) four-quark states are possible. Under the hypothesis that the physical hadron states are colorless, this implies that we should not observe any particles made of four quarks.

5.4 Exercises

1. How many transpositions are required to generate a k-cycle? Hence, what is the parity of a k-cycle?

2. Show that, for any $P_a \in S_n$:

$$\begin{aligned} P_a S &= S \\ P_a A &= A P_a = \delta_{P_a} A \\ S^2 &= S \\ A^2 &= A. \end{aligned}$$

3. We gave the representation of one element of the two-dimensional irreducible representation of S_3 in basis $\{\psi'_1, \psi'_2\}$ in Eqn. 5.38. Find the other matrices in this representation.

4. Quarks are spin-$\frac{1}{2}$ particles, hence they are fermions. According to quantum mechanics , the wave function of a system of identical fermions must be antisymmetric under the interchange of the fermions (the celebrated "connection between spin and statistics"). To see the idea, first consider a system of two electrons (an electron is also a spin-$\frac{1}{2}$ particle). We put the "first" electron at position x_1, with spin orientation s_1, and the second at x_2 with spin orientation s_2. The wave function is $\psi(x_1, s_1; x_2, s_2)$. This wave function must be antisymmetric under interchange of the two electrons:
$$\psi(x_2, s_2; x_1, s_1) = -\psi(x_1, s_1; x_2, s_2). \tag{5.72}$$

   Suppose our two electrons are in an orbital angular momentum $L = 0$ state. The spin states may be described by the z components of the spins, $\pm\frac{1}{2}$, which we'll represent with arrows, $\uparrow$ for spin "up" and $\downarrow$ for spin "down". But in making a system of two electrons (with $L = 0$), we are generating a product representation of $SU(2)$ in angular momentum: $2 \otimes 2 = 3 \oplus 1$. That is the irreducible representations of our total angular momentum state are three-dimensional, corresponding to total spin one, and one-dimensional or spin zero. We have already worked out the symmetries of these combinations in this note: the spin one system is symmetric under interchange, and the spin zero is antisymmetric. Note that, since we have specified $L = 0$ the wave function is symmetric under the interchange of the spatial coordinates. We may conclude that the only way we can put two electrons together in an $L = 0$ state is with total spin $S = 0$:
$$\psi(x_2, s_2; x_1, s_1) = \frac{1}{\sqrt{2}}(|e\uparrow; e\downarrow\rangle - |e\downarrow; e\uparrow\rangle), \tag{5.73}$$
   where the symmetry under spatial interchange is not explicity shown.

   Now let us return to quarks, and consider baryons. To keep this simple, we'll also put our three quarks together in a state with no orbital angular momentum (S-wave). That is, the spatial state is symmetric under the interchange of any pair of quarks. We'll regard the "flavor" quantum number ("u", "d", or "s", or equivalently, I_3, Y) as analogous to the spin projections, and regard them as additional quantum numbers that get interchanged when we act on a wave function with permutations of the quarks.

   Treating the angular momentum, when we combine three quarks in S-wave, we build the $2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$ representation of $SU(2)$. Thus, the three quarks could be in a total spin state of 1/2 or 3/2. The spin 3/2 state is clearly symmetric under interchange of the spins. The two spin 1/2 representations have mixed symmetry. We may chose a basis for one of these representations that corresponds to symmetry under the interchange of the quarks at x_1 and x_2 (or, quarks 1 and 2, for short):

$$\begin{aligned} \chi^\lambda_+ &= -\frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \\ \chi^\lambda_- &= -\frac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow). \end{aligned} \tag{5.74}$$

Likewise, the spin basis wavefunctions for the other two dimension wave function, with antisymmetry under interchange of the first two quarks, may be chosen as:

$$\begin{aligned} \chi^\rho_+ &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ \chi^\rho_- &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow). \end{aligned} \tag{5.75}$$

We must deal with a small (but extremely important in physical implication!) complication before we construct the (spin, flavor) wave functions of the S-wave baryons. Consider the Δ^{++} baryon. This is made of three u quarks, clearly in a symmetric flavor state. It is also a spin-$\frac{3}{2}$ particle, with all of the quark spins aligned, that is, in a spin symmetric state. Thus, the Δ^{++} is symmetric in spatial interchange (since it is S-wave), flavor interchange, and spin interchange. Combined, it appears that we have built a baryon which is symmetric under interchange of the constituent quarks. But this violates our fermion principle, which says it must be antisymmetric. This observation was historically one of the puzzles in the 1960's when this model was proposed. Eventually, we learned that the most promising way out was to give the quarks another quantum number, called "color". To combine three quarks with three different colors requires a minimum of three colors, hence the hypothesis that there are three colors, and the relevant group for rotations in color space is also the $SU(3)$ group. It is a hypothesis (perhaps justifiable with QCD) that the physical particles (such as baryons) we see are overall colorless. That is, the color basis wave function corresponds to a singlet representation of $SU(3)_{\text{color}}$. We have already seen that the one-dimensional representation in the decompostion $3\otimes 3\otimes 3 = 10\oplus 8\oplus 8\oplus 1$ is antisymmetric under interchange. Thus, the introduction of color saves our fermi statistics. We simply assume that the color wavefunction of baryons is antisymmetric. Then the (space, spin, flavor) wave function must be overall symmetric.

Now consider the proton, a spin $\frac{1}{2}$ baryon made with two u's and a d. In $SU(3)_{\text{flavor}}$, the proton wave function must be some linear combination of the basis states corresponding to the $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ representations (you may wish to ponder why there is no $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ piece). Let us call the (12)-symmetric wave functions ϕ^λ and the (12)-antisymmetric wave functions ϕ^ρ:

$$\phi^{\lambda}_{uud} = -\frac{1}{\sqrt{6}}(udu + duu - 2uud) \tag{5.76}$$

$$\phi^{\rho}_{uud} = \frac{1}{\sqrt{2}}(udu - duu). \tag{5.77}$$

The problem you are asked to solve is: What is the wave function of a spin up proton? Assume that the spatial wave function is symmetric, and give the spin/flavor wave function. It is perhaps easiest to use some notation such as kets, forming the wave function from kets of the form $|u\uparrow u\uparrow d\downarrow\rangle$, etc.

Note: I won't go into the physics further, but it should be remarked that this isn't just an idle exercise in mathematics – this wave function implies observable physical consequences on quantities such as the magnetic moment of the proton.