

Physics 129a
Integral Equations
Solutions to Problems
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1 Exercises

1. Given an abstract complex vector space (linear space), upon which we have defined a scalar product (inner product):

$$\langle a|b \rangle \tag{1}$$

between any two vectors a and b , prove the Schwarz inequality:

$$|\langle a|b \rangle|^2 \leq \langle a|a \rangle \langle b|b \rangle. \tag{2}$$

Give the condition for equality to hold.

One way to approach the proof is to consider the fact that the projection of a onto the subspace which is orthogonal to b cannot have a negative length, where we define the length (norm) of a vector according to:

$$\|c\| \equiv \sqrt{\langle c|c \rangle}. \tag{3}$$

Further, prove the triangle inequality:

$$\|a + b\| \leq \|a\| + \|b\|. \tag{4}$$

2. Considering our RC circuit example, derive the results in Eqn. ?? through Eqn. ?? using the Fourier transform.
3. Prove the convolution theorem.
4. We showed the the Fourier transform of a Gaussian was also a Gaussian shape. That is, let us denote a Gaussian of mean μ and standard deviation σ by:

$$N(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right]. \tag{5}$$

- (a) In class we found that the Fourier Transform of a Gaussian of mean zero was:

$$\hat{N}(y; 0, \sigma) = \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{y^2 \sigma^2}{2} \right]. \quad (6)$$

Generalize this result to find the Fourier transform of $N(x; \mu, \sigma)$.

- (b) The experimental resolution function of many measurements is approximately Gaussian in shape (third quarter you'll prove the "Central Limit Theorem"). Often, there is more than one source of uncertainty contributing to the final result. For example, we might measure a distance in two independent pieces, with means μ_1, μ_2 and standard deviations σ_1, σ_2 . The resolution function (sampling distribution) of the final result is then the convolution of the two pieces:

$$P(x; \mu_1, \sigma_1, \mu_2, \sigma_2) = \int_{-\infty}^{\infty} N(y; \mu_1, \sigma_1) N(x - y; \mu_2, \sigma_2) dy. \quad (7)$$

Do this integral to find $P(x; \mu_1, \sigma_1, \mu_2, \sigma_2)$. Note that it is possible to do so by straightforward means, though it is a bit tedious. You are asked here to instead use Fourier transforms to (I hope!) obtain the result much more easily.

5. The "Gaussian integral" is:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right] dx = 1. \quad (8)$$

Normally, the constants μ and σ^2 are real. However, we have encountered situations where they are complex. Is this integral valid for arbitrary complex (including pure imaginary) μ and σ^2 ? Try to do a very careful and convincing demonstration of your answer.

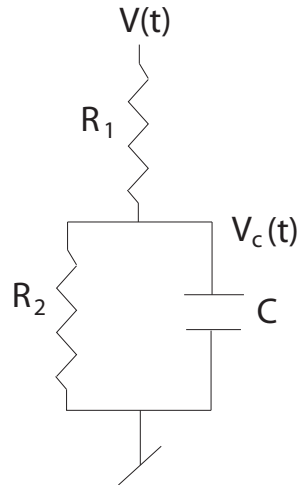
6. In class we consider the three-dimensional Fourier transform of $e^{-\mu r}/r$, where $r = \sqrt{x^2 + y^2 + z^2}$. What would the Fourier transform be in two dimensions (i.e., in a two-dimensional space with $r = \sqrt{x^2 + y^2}$)?
7. The lowest P -wave hydrogen wave function in position space may be written:

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{32\pi a_0^5}} r \cos \theta \exp \left(-\frac{r}{2a_0} \right), \quad (9)$$

where $r = \sqrt{x^2 + y^2 + z^2}$, θ is the polar angle with respect to the z axis, and a_0 is a constant. Find the momentum-space wave function for this state (i.e., find the Fourier transform of this function).

In this and all problems in this course, I urge you to avoid look-up tables (e.g., of integrals). If you do feel the need to resort to tables, however, be sure to state your source.

8. In section ??, we applied the Laplace transform method to determine the response of the RC circuit:



to an input voltage $V(t)$ which was a delta function. Now determine $V_C(t)$ for a pulse input. Model the pulse as the difference between two exponentials:

$$V(t) = A \left(e^{-t/\tau_1} - e^{-t/\tau_2} \right). \quad (10)$$

9. In considering the homogeneous integral equation, we stated the theorem that there are a finite number of eigenfunctions for any given eigenvalue. We proved this for real functions; now generalize the proof to complex functions.

Solution: Proof: We'll repeat the proof in the note, but now allowing for complex functions. Given an eigenfunction f_j corresponding to eigenvalue λ , let:

$$p_j(x) \equiv \int_a^b k(x, y) f_j(y) dy = \frac{1}{\lambda} f_j(x). \quad (11)$$

Now consider, for some set of n eigenfunctions corresponding to eigenvalue λ :

$$D(x) \equiv |\lambda|^2 \int_a^b \left| k(x, y) - \sum_{j=1}^n p_j^*(x) f_j(y) \right|^2 dy. \quad (12)$$

It must be that $D(x) \geq 0$ because the integrand is nowhere negative for any x . Note that the sum term may be regarded as an approximation to the kernel, hence $D(x)$ is a measure of the closeness of the approximation. With some manipulation:

$$D(x) = |\lambda|^2 \int_a^b |k(x, y)|^2 dy - |\lambda|^2 \int_a^b \sum_{j=1}^n \left[k(x, y) p_j^*(x) f_j(y) + k^*(x, y) p_j(x) f^*(y) \right] dy$$

$$\begin{aligned}
& + |\lambda|^2 \int_a^b \left| \sum_{j=1}^n p_j^*(x) f_j(y) \right|^2 dy \\
= & |\lambda|^2 \int_a^b |k(x, y)|^2 dy - 2|\lambda|^2 \sum_{j=1}^n |p_j(x)|^2 \\
& + |\lambda|^2 \sum_{j=1}^n p_j^*(x) \sum_{k=1}^n p_k(x) \int_a^b f_j(y) f_k^*(y) dy \\
= & |\lambda|^2 \int_a^b [k(x, y)]^2 dy - |\lambda|^2 \sum_{j=1}^n |p_j(x)|^2.
\end{aligned} \tag{13}$$

With $D(x) \geq 0$, we have thus proved a form of Bessel's inequality. We may rewrite the inequality as:

$$|\lambda|^2 \int_a^b |k(x, y)|^2 dy \geq \sum_{j=1}^n |f_j(x)|^2. \tag{14}$$

If we integrate both sides over x , we obtain:

$$\begin{aligned}
|\lambda|^2 \int_a^b \int_a^b |k(x, y)|^2 dy dx & \geq \sum_{j=1}^n \int_a^b |f_j(x)|^2 dx \\
& \geq n,
\end{aligned} \tag{15}$$

using the normalization of the f_j . As long as $\int \int |k|^2 dx dy$ is bounded, we see that n must be finite. For finite a and b , this is certainly satisfied, by our continuity assumption for k . Otherwise, we may impose this as a requirement on the kernel.

10. Give a graphical proof that the series $D(\lambda)$ and $D(x, y; \lambda)$ in the Fredholm solution are polynomials of degree n if the kernel is of the degenerate form:

$$k(x, y) = \sum_{i=1}^n \phi_i(x) \psi_i(y). \tag{16}$$

11. Solve the following equation for $u(t)$:

$$\frac{d^2 u}{dt^2}(t) + \int_0^1 \sin[k(s-t)] u(s) ds = a(t), \tag{17}$$

with boundary condition $u(0) = u'(0) = 0$, and $a(t)$ is a given function.

12. Prove that an n -term degenerate kernel possesses at most n distinct eigenvalues.

13. Solve the integral equation:

$$f(x) = e^x + \int_1^x \frac{1+y}{x} f(y) dy. \tag{18}$$

Hint: If you need help solving a differential equation, have a look at Mathews and Walker chapter 1.

14. In section ?? we developed an algorithm for the numerical solution of Volterra's equation. Apply this method to the equation:

$$f(x) = x + \int_0^x e^{-xy} f(y) dy. \quad (19)$$

In particular, estimate $f(1)$, using one, two, and three intervals (i.e., $N = 1$, $N = 2$, and $N = 3$). [We're only doing some low values so you don't have to develop a lot of technology to do the computation, but going to high enough N to get a glimpse at the convergence.]

15. Another method we discussed in section ?? is the extension to the Laplace transform in Laplace's method for solving differential equations. I'll summarize here: We are given a differential equation of the form:

$$\sum_{k=0}^n (a_k + b_k x) f^{(k)}(x) = 0 \quad (20)$$

We assume a solution of the form:

$$f(x) = \int_C F(s) e^{sx} ds, \quad (21)$$

where C is chosen depending on the problem. Letting

$$U(s) = \sum_{k=0}^n a_k s^k \quad (22)$$

$$V(s) = \sum_{k=0}^n b_k s^k, \quad (23)$$

the formal solution for $F(s)$ is:

$$F(s) = \frac{A}{V(s)} \exp \int^s \frac{U(s')}{V(s')} ds', \quad (24)$$

where A is an arbitrary constant.

A differential equation that arises in the study of the hydrogen atom is the Laguerre equation:

$$x f''(x) + (1 - x) f'(x) + \lambda f(x) = 0. \quad (25)$$

Let us attack the solution to this equation using Laplace's method.

- (a) Find $F(s)$ for this differential equation.
- (b) Suppose that $\lambda = n = 0, 1, 2, \dots$. Pick an appropriate contour, and determine $f_n(x)$.

16. Write the diagram, with coefficients, for the fifth-order numerator and denominator of the Fredholm expansion.

17. Solve the equation:

$$f(x) = \sin x + \lambda \int_0^\pi \cos x \sin y f(y) dy \quad (26)$$

for $f(x)$. Find any eigenvalues and the corresponding eigenfunctions. Hint: This problem is trivial!

18. Find the eigenvalues and eigenfunctions of the kernel:

$$\begin{aligned} k(x, y) &= \frac{1}{2} \log \left| \sin \frac{x+y}{2} / \sin \frac{x-y}{2} \right| \\ &= \sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n}, \quad 0 \leq x, y \leq \pi. \end{aligned} \quad (27)$$

19. In the notes we considered the kernel:

$$k(x, y) = \frac{1}{2\pi} \frac{1 - \alpha^2}{1 - 2\alpha \cos(x - y) + \alpha^2}, \quad (28)$$

where $|\alpha| < 1$ and $0 \leq x, y \leq 2\pi$. Solve the integral equation

$$f(x) = e^x + \lambda \int_0^{2\pi} k(x, y) f(y) dy \quad (29)$$

with this kernel. What happens if λ is an eigenvalue? If your solution is in the form of a series, does it converge?

20. Solve for $f(x)$:

$$f(x) = x + \int_0^x (y - x) f(y) dy, \quad (30)$$

This problem can be done in various ways. If you happen to obtain a series solution, be sure to sum the series.

21. We wish to solve the following integral equation for $f(x)$:

$$f(x) = g(x) - \lambda \int_0^x f(y) dy, \quad (31)$$

where $g(x)$ is a known, real continuous function with continuous first derivative, and satisfies $g(0) = 0$.

- (a) Show that this problem may be re-expressed as a differential equation with suitable boundary condition, which may be written in operator form as $Lf = g'$. Give L explicitly and show that it is a linear operator.

- (b) Suppose that $G(x, y)$ is the solution of $LG = \delta(x - y)$, where $\delta(x)$ is the Dirac δ function. Express the solution to the original problem in the form of an integral transform involving G and g' .
- (c) Find $G(x, y)$ and write down the solution for $f(x)$.
22. Some more Volterra's equations: Solve for $f(x)$ in the following two cases –
- (a) $f(x) = \sin x + \cos x + \int_0^x \sin(x - y)f(y) dy,$
- (b) $f(x) = e^{-x} + 2x + \int_0^x e^{y-x}f(y) dy.$

23. Consider the following LCR circuit:

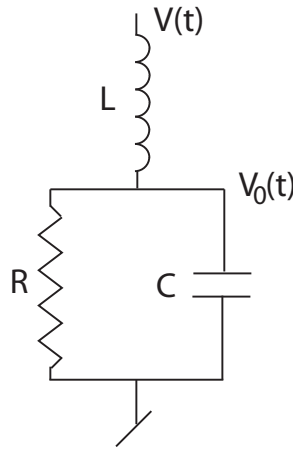


Figure 1: An LCR circuit.

Use the Laplace transform to determine $V_0(t)$ given

$$V(t) = \begin{cases} V & 0 < t < T \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

Make a sketch of $V(t)$ for (a) $2RC > \sqrt{LC}$; (b) $2RC < \sqrt{LC}$; (c) $2RC = \sqrt{LC}$.