Physics 129a
Measure Theory
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1 Introduction

The rigorous mathematical underpinning of much of what we have discussed, such as the construction of a proper Hilbert space of functions, is to be found in "measure theory". We thus refine the concepts in our note on Hilbert spaces here. It should immediately be stated that the term "measure" refers to the notion of measuring the "size" of a set. This is the subject of measure theory. With measure theory, we will find that we can generalize the Riemann notion of an integral. [This note contains all the essential ideas to complete the development begun in the note on Hilbert spaces for the construction of a suitable Hilbert space for quantum mechanics. However, this note is still under consruction, as there remain gaps in the presentation.]

Let us motivate the discussion by considering the space, $C_2(-1,1)$ of complex-valued continuous functions on $[-1,1]$. We define the norm, for any $f(x) \in C_2[-1,1]$:

$$||f||^2 = \int_{-1}^{1} |f(x)|^2 dx. \tag{1}$$

Consider the following sequence of functions, $f_1, f_2, \ldots$, in $C_2[-1,1]$:

$$f_n(x) = \begin{cases} -1 & -1 \le x \le -1/n, \\ nt & -1/n \le x \le 1/n, \\ 1 & 1/n \le x \le 1. \end{cases} \tag{2}$$

Fig. 1 illustrates the first few of these functions. This set of functions defines a Cauchy sequence, with convergence to the discontinuous function

$$f(x) = \begin{cases} -1 & -1 \le x \le 0, \\ 1 & 0 < x \le 1. \end{cases} \tag{3}$$

Since $f(x) \notin C_2[-1,1]$, this space is not complete.

How can we "complete" such a space? We need to add discontinuous functions somehow. Can we simply add all piecewsie continuous functions? Consider the sequence of piecewise continuous functions, $g_1, g_2, \ldots$:

$$g_n(x) = \begin{cases} 0 & -1 \le x < -1/n, \\ 1 & -1/n \le x \le 1/n, \\ 0 & 1/n < x \le 1. \end{cases} \tag{4}$$

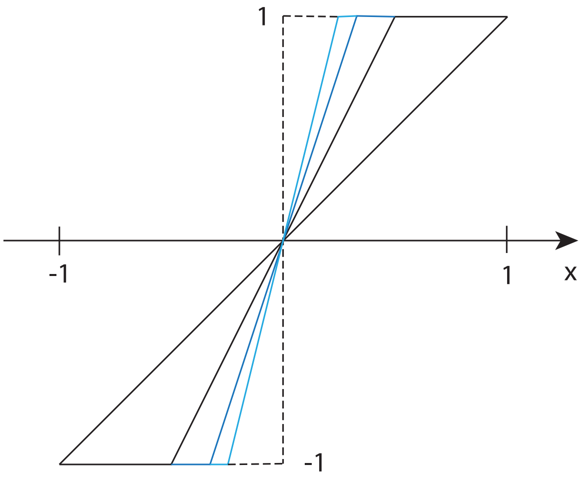

Figure 1: A Cauchy sequence of functions in $C_2[-1,1]$.

This sequence is illustrated in Fig. 1. This sequence converges to the function equal to 1 at $x = 0$, and zero everywhere else. This isn't exactly what we have in mind when we say "piecewise discontinuous", but perhaps it is all right. However, the sequence also converges to $f(x) = 0$. This gives us two functions in our vector space with norm zero, which is not allowed. We need to think some more if we are going to lay a rigorous foundation. Measure theory will permit us to deal with these issues.

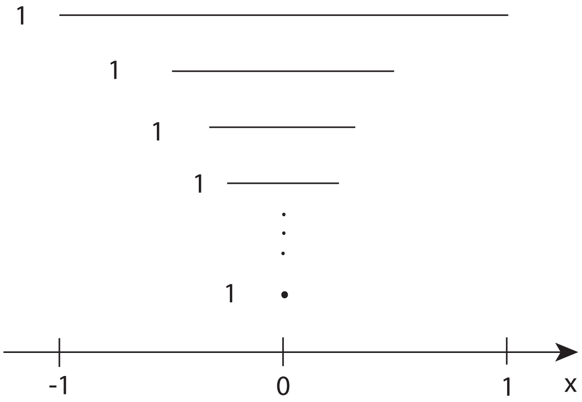

Figure 2: A Cauchy sequence of piecewise discontinuous functions. The coordinate system is moved for each function in order to separate them vertically.

2 The Jordan Measure

Consider the set, $\mathcal{R}^n$, of n-tuples of real numbers. We may define a generalization of the closed interval on the real numbers as the "generalized interval":

$$I(a,b) \equiv \{x : x \in \mathcal{R}^n, \text{ with components } a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}. \quad (5)$$

We define a familiar measure of the "size" of the set $I(a,b)$:

$$\text{measure}\,[I(a,b)] = \prod_{i=1}^{n}(b_i - a_i). \quad (6)$$

Let us use this idea to define the measure of more general subsets of $\mathcal{R}^n$. Another notion of size is the "diagonal":

$$\Delta(a,b) \equiv \sqrt{\sum_{i=1}^{n}(b_i - a_i)^2}. \quad (7)$$

Suppose B is a bounded subset of $\mathcal{R}^n$. We wish to find a suitable measure of B, consistent with our measure of a generalized interval, and reflecting our usual notion of a "volume" in n dimensional $\mathcal{R}^n$. We begin by choosing a generalized interval, I, containing B. It will be sufficient to consider coverings of I, since they also contain all points of B.

Construct a covering of I from a finite number of generalized intervals, no two containing interior points in common. Note that this is possible, since I itself is a generalized interval, and we could imagine dividing it in half in some dimension, and we can then divide the two halves in some dimension, etc. See Fig. 2 for an illustration. Out of this set of generalized intervals, some may contain one or more points of the closure of B, $\bar{B}$. Call these generalized intervals $B_1, B_2, \ldots, B_n$. There may also be elements of our set of generalized intervals which consist only of interior points of B. Call these $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_m$. It may happen that $m = 0$. Note that $\{\tilde{B}_i\} \subset \{B_i\}$.

Now form the sums:

$$S = \sum_{i=1}^{n}\text{measure}B_i, \quad \tilde{S} = \sum_{i=1}^{m}\text{measure}\tilde{B}_i. \quad (8)$$

Let J_o (for "outer Jordan measure") be the greatest lower bound on the set of possible sums S, and J_i (for "inner Jordan measure") be the least upper bound on $\tilde{S}$. Also, let Δ be the greatest diagonal of the intervals $B_1, B_2, \ldots, B_n$. Considering possible coverings, we have

Theorem:

$$\lim_{\Delta\to 0} S = J_o, \tag{9}$$

$$\lim_{\Delta\to 0} \tilde{S} = J_i. \tag{10}$$

If $J_o = J_i$ then set B is said to be Jordan measurable, with

$$\text{measure } B = J_o = J_i. \tag{11}$$

We may investigate the properties of the Jordan measure. Suppose $M_1, \ldots, M_m$ are m disjoint (Jordan) measurable sets. Then the set

$$\cup_{i=1}^m M_i \tag{12}$$

is measurable, with

$$\text{measure}\,(\cup_{i=1}^m M_i) = \sum_{i=1}^m \text{measure } M_i. \tag{13}$$

That is, the Jordan measure is additive.

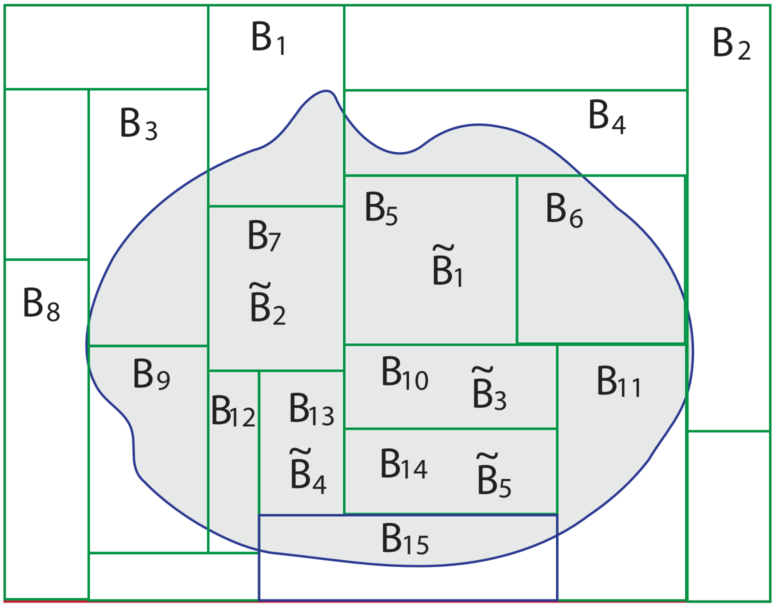

Figure 3: Construction of the Jordan measure.

However, this additivity may not hold if $m = \infty$ – the Jordan measure is not denumerably additive. For example, suppose $B = \{\text{rational numbers between 0 and 1}\}$. This is the union of a denumerable family of disjoint sets, R_i, each set consisting of a single number. The measure of each R_i is zero. The outer Jordan measure of B is one, since every finite partition of $[0, 1]$ by generalized intervals that contains all points in B also contains all irrational numbers. On the other hand, the inner Jordan meausre of B is zero, since there are no sets consisting only of interior points of B (that is, B has no interior points). Since $J_o \neq J_i$, this set is not Jordan measurable.

3 The Lebesgue Measure

It is possible to generalize our Jordan measure such that we can measure the sizes of more sets, including $B = \{\text{rational numbers between 0 and 1}\}$ and sets in spaces other than $\mathcal{R}^n$. The result is the Lebesgue measure. The construction has similarities with the Jordan measure, but is more abstract.

We begin with the following abstraction of the notion of the generalized interval:

Definition (*Semi-ring*): Let S be a non-empty set and let Σ be a family of subsets of S satisfying:

1. $\emptyset \in \Sigma$;
2. if $A \in \Sigma$ and $B \in \Sigma$, then $A \cap B \in \Sigma$;
3. if $A \in \Sigma$ and $B \in \Sigma$, with $B \subset A$, then $\exists$ a family of n disjoint sets $B_1, B_2, \ldots, B_n \in \Sigma$ such that

$$A - B = \cup_{i=1}^{\infty} B_i, \tag{14}$$

then Σ is a **semi-ring** of subsets of S. Note that the notation $A - B$ means the smallest set that can be added to B to get A. Another way of writing it is $A \cap \tilde{B}$, where $\tilde{B}$ is the complement of B in S.

For example, consider the family of "semi-closed generalized intervals" in $\mathcal{R}^n$:

$$\Sigma_n \equiv \{x : a_i \leq x_i < b_i, i = 1, 2, \ldots, n\}. \tag{15}$$

This family is a semi-ring. For exmple, let's consider Σ_2, and check against the properties of a semi-ring:

- With $b_i = a_i$, $\emptyset \in \Sigma_2$.
- Fig. 3a suggests that if $A, B \in \Sigma_2$, then $A \cap B \in \Sigma_2$.

- Fig. 3b suggests that if $A, B \in \Sigma_2$, and $B \subset A$, then $A - B = B_1 \cup B_2 \cup B_3$, where $B_1, B_2, B_3 \in \Sigma_2$, and B_1, B_2, B_3 are disjoint.

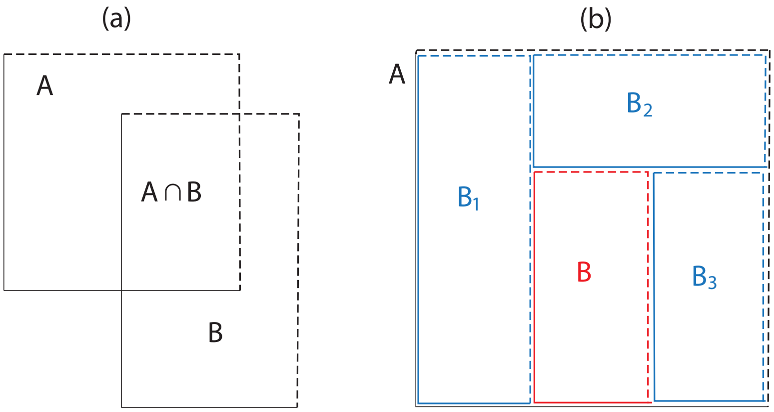

Figure 4: Illustration of possible semi-closed generalized intervals in $\mathcal{R}^2$.

Given our abstract notion of an "interval" as elements of a semi-ring, we proceed to define the notion of a measure on the semi-ring.
Definition (*Measure on a semi-ring*): Let $\mu : \Sigma \to \mathcal{R}$ be a mapping from a semi-ring to non-negative real numbers. This mapping defines a **measure** on Σ iff:

- If $A, B \in \Sigma$ and $B \subset A$ then $\mu(A) \geq \mu(B)$.
- If $A = \cup_{i=1}^{\infty} \in \Sigma$ and $A_i \in \Sigma$ for $i = 1, 2, 3, \ldots$ with $A_i \cap_{i \neq j} A_j = \emptyset$, then $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ (denumerable addititivty).
- $\mu(\emptyset) = 0$.

We may consider some examples:

- Consider the semi-ring formed by the family of all sets of particles in the universe (assume this is a well-defined concept). The function:

$$\mu(A) = \sum_{\text{particles in set } A} m_{\text{particle}}, \tag{16}$$

  where m_{particle} is the rest mass of the particle, defines a measure on this semi-ring.

- With the semi-ring generated by semi-closed general intervals:

  $$A = \{x : a_i \leq x_i < b_i; i = 1, 2, \dots, n; x \in \mathcal{R}^n\}, \tag{17}$$

  the mapping

  $$\mu(A) = \prod_{i=1}^{n} (b_i - a_i) \tag{18}$$

  defines a measure on this semi-ring.

- Let S be an arbitrary space, with $x_0 \in S$. Let Σ be the semi-ring consisting of all subsets of S. For every element $A \in \Sigma$ define:

  $$\mu(A) = \begin{cases} 1 & \text{if } s_0 \in A \\ 0 & \text{if } s_0 \notin A. \end{cases} \tag{19}$$

  The mapping μ defines a measure, called the Dirac measure.

We have defined a measure μ on a semi-ring Σ in space S, which we may denote as the triplet (S, Σ, μ). Just like we extended the notion of our measure of the generalized interval in $\mathcal{R}^n$ to other subsets of $\mathcal{R}^n$ with the Jordan measure, we would like to extend our abstract measure to subsets of S that do not belong to the semi-ring. As in the case of the Jordan measure, we approximate our set with elements of the semi-ring:

Definition (*Outer measure*): Given (S, Σ, μ) and set $A \subset S$. Either:

- A denumerable family of sets $A_i \in \Sigma$ such that $A \subset \cup_{i=1}^{\infty} A_i$ does not exist. Then we set **outer measure** $\mu^*(A) = \infty$.

- There exists at least one family of sets $A_i \in \Sigma$ such that $A \subset \cup_{i=1}^{\infty} A_i$. Taking the set of numbers

  $$\sum_{i=1}^{\infty} \mu(A_i) \tag{20}$$

  for all sequences of sets $\{A_i\}$ such that $A \subset \cup_{i=1}^{\infty} A_i$, we define the **outer measure**

  $$\mu^*(A) = \inf \left[\sum_{i=1}^{\infty} \mu(A_i) \right]. \tag{21}$$

We have the following theorem that states that the outer measure fulfills several desirable properties for the notion of a size:

Theorem: The outer measure satisfies:

- If $A \in \Sigma$, then $\mu^*(A) = \mu(A)$.

- Subadditivity of μ^*: $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$.
- If $B \subset A$ then $\mu^*(A) \geq \mu^*(B)$.

Proof: Let's prove the first part. If $A \in \Sigma$, then there exists a denumerable family of subsets in Σ such that A is contained in their union. In particular, the set A alone satisfies this. Since A is the smallest such family of subsets, $\mu^*(A) = \mu(A)$.

But we'd really like more, especially the denumerable additivity property so we can deal with issues of infinity. We may achieve this as follows; first define a notion of measurability:

Definition (*Measurable sets*): A set A in space S is called **measurable** iff:

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap A) + \mu^*(T \cap \tilde{A}), \\ &= \mu^*(T \cap A) + \mu^*(T - A), \quad \forall T \subset S. \end{aligned} \tag{22}$$

In other words, a set is measurable if it partitions any set T into two subsets such that the outer measure of these two subsets is additive.

Next we define a "σ-ring:

Definition (σ*-ring*): Let Λ be a family of subsets of space S such that:

1. $\emptyset \in \Lambda$.
2. If $A \in \Lambda$, then the complement is also: $\tilde{A} \in \Lambda$.
3. If $A_1, A_2, \ldots \in \Lambda$, then their union is also: $\cup_{i=1}^{\infty} A_i \in \Lambda$.

Then Λ is called a **σ-ring** on S.

A σ-ring is also a semi-ring:

1. $\emptyset \in \Lambda$.
2. If $A \in \Lambda$ and $B \in \Lambda$, then $\tilde{A} \in \Lambda$, $\tilde{B} \in \Lambda$ and $\tilde{A} \cup \tilde{B} \in \Lambda$. Hence, $A \cap B \in \Sigma = -(\tilde{A} \cup \tilde{B} \in \Lambda) \in \Lambda$.
3. If $A \in \Sigma$ and $B \in \Sigma$, with $B \subset A$, then $A - B = A \cap \tilde{B} \in \Lambda$, as follows from the previous line. Hence, $\exists$ a family of n disjoint sets $B_1, B_2, \ldots, B_n \in \Sigma$ such that

$$A - B = \cup_{i=1}^{\infty} B_i, \text{ in particular } B_1 = A - B \in \Lambda. \tag{23}$$

A simple example of a σ-ring is the family of all subsets of S.

We don't really need the concept of a "ring" here, but mention for the curious that the σ-ring is a generalization of a ring. For a ring, only finite unions are required to be elements.

Now for the desired theorem that gives us denumerable additivity:

Theorem: Let Λ be the family of all measurable sets in S. Then Λ is a σ-ring, and outer measure μ^* is a denumerably additive set function defined on Λ.

Proof: If A is measurable, then so is $\tilde{A}$, i.e., if $A \in \Lambda$ then $\tilde{A} \in \Lambda$. We show that $\emptyset \in \Lambda$ as follows:

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap \emptyset) + \mu^*(A \cap \tilde{\emptyset}) && (24) \\ &= 0 + \mu^*(A). && (25) \end{aligned}$$

Thus, $\emptyset \in \Lambda$. It remains to demonstrate that if a set of subsets $A_1, A_2, \ldots$ is measurable, then their countable union is measurable.

We consider now the construction of a measure on the real numbers according to these ideas. Let Λ be the family of measurable sets on $\mathcal{R}$, with the outer measure μ^* defined as the extension of the measure $\mu(A) = b - a$ for elements $A \in \Lambda$ given by sets of the form $\{x : a \leq x < b\}$. This is called the "Lebesgue measure". We are going to drop the adjective "outer" from now on, and just write μ for μ^*.

Let $\{p_0\}$ be a set consisting of a single point p_0. Is $\{p_0\}$ measurable? To investigate, write

$$\{p_0\} = \cap_{n=1}^{\infty} I_n(p_0), \tag{26}$$

where $I_n(p_0)$ are the intervals

$$I_n(p_0) \equiv \{x : p_0 \leq x < p_0 + 1/n\}. \tag{27}$$

Since $I_n \in \Lambda$ and Λ is a σ-ring, we have

$$\cap_{n=1}^{\infty} I_n(p_0) = - \cup_{i=1}^{\infty} \tilde{I}_n \in \Lambda. \tag{28}$$

Thus, a single point is an element of the σ-ring, and has measure zero. That its measure is zero follows from the facts that $\mu(\{p_0\}) \leq \mu(I_n), n = 1, 2, \ldots$, and $\lim_{n\to\infty} \mu(I_n) = 0$.

We see now that the set of rational numbers in $(0, 1)$ is a measurable set, with measure $\sum_{i=1}^{\infty} 0 = 0$, since the rationals are a denumberable set. It should also be remarked that, where the Jordan measure exists, the Lebesgue and Jordan measures are equal.

We may readily extend the Lebesgue measure to $\mathcal{R}^n$:

Theorem: Open sets, and hence also closed sets, in the usual topology on $\mathcal{R}^n$ are Lebesgue measurable.

Definition (*Borel set*): The minimal family of sets, Λ_B, that is a σ-ring and that contains all open subsets of $\mathcal{R}^n$ is called the Borel σ-ring. The sets belonging to it are called Borel sets.

A natural question is whether every subset of $\mathcal{R}^n$ is measurable. The answer is no, making use of the axiom of choice (in the form of Zermelo's axiom: For an arbitrary family of non-empty sets there exists a mapping from each set into one of its elements).

4 Function spaces

Let us now apply our measure theory to function spaces. This will permit a generalized notion of the integral, as well as defining a suitable Hilbert space for quantum mechanics.

We start with the notion of a "measurable function":

Definition (*Measurable function*): Let $f(s)$ be a real function defined on space S. This function is called **measurable** iff, for every real number ρ, the set

$$S_\rho = \{s : f(s) < \rho, \text{ where } s \in S\} \tag{29}$$

is measurable. Fig. 4 illustrates this concept.

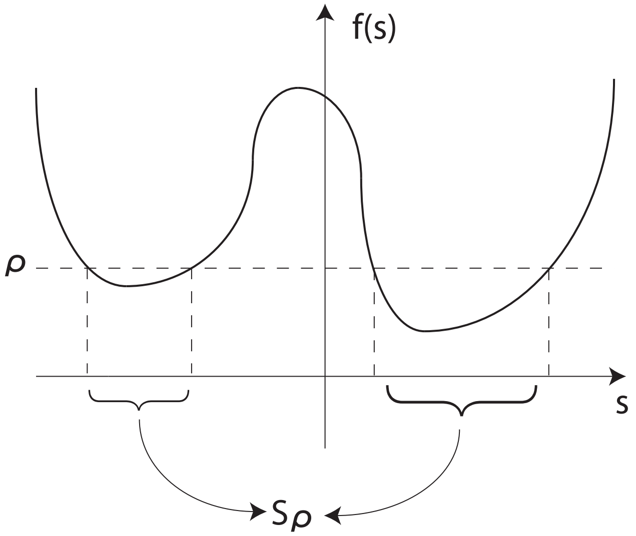

Figure 5: Illustration of the definition of a measurable function.

Theorem: If $f(y)$, for $y \in \mathcal{R}$, is a continuous, real-valued function, and if function $g(s)$ on space S is measurable, then $f[g(s)]$ is also a measurable function on S.

An immediate consequence of this teorem is that continuous functions are measurable, since we could take $g(s) = s$ on $S = \mathcal{R}$.

We also have that

Theorem: If function $f(s)$ is measurable, then the set

$$A_\rho = \{s : f(s) = \rho, s \in S\} \tag{30}$$

is measurable.

That is, the set whose image under a measurable function is a given value is a measureable set. The proof of these two theorems relies on the fact that a real function on S is measurable iff the inverse image of every Borel set is measurable. The reader is referred to other sources for development of this.
Definition (*Characteristic function*): Let $A \subset S$. The function

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \notin A \end{cases} \tag{31}$$

is called the **characteristic function** of set A. We remark that χ_A is a measurable function iff A is a measurable set.[1]
Definition (*μ-simple function*): A function of the form

$$f(s) = \sum_{n-1}^{\infty} a_n \chi_{A_n}(s), \tag{32}$$

where $s \in S$, the a_n are arbitrary real numbers, and the sets A_n are disjoint measurable subsets of S such that $\cup_n A_n = S$, is called a **μ-simple function**. For example, a function of the following form is μ-simple (for disjoint measurable A_1, A_2):

$$\begin{aligned} f(s) &= a_1 \chi_{A_1}(s) + a_2 \chi_{a_2}(s) && (33) \\ &= \begin{cases} a_1 & s \in A_1 \\ a_2 & s \in A_2 \\ 0 & s \in -(A_1 \cup A_2). \end{cases} && (34) \end{aligned}$$

The values of $f(s)$ are given by the sequence $\{a_n\}$, where we may presume that the values are distinct; otherwise if $f(s) = a_1\chi_{A_1} + a_1\chi_{A_2} + \ldots$, we can write $f(s) = a_1\chi_{A_1 \cup A_2} + \ldots$ It is readily demonstrated that the set of μ-simple functions defines a linear space.

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[1] Note that this definition of a characteristic function has nothing to do with the notion of a characteristic function in probability.

Theorem: A μ-simple function is measurable.

Proof: Let f be a μ-simple function. Consider set $R_\rho \equiv \{x : x < \rho\}$. Since $f^{-1}(R_\rho)$ is the union of an at most denumerable family of sets A_n such that $a_n < \rho$ and the A_n are measurable, then $f^{-1}(R_\rho)$ is measurable. Hence f is measurable.

The following theorem tells us that μ-simple functions are "enough".

Theorem: The limit of a pointwise convergent sequence of measurable functions is a measurable function. A function is measurable iff it is the limit of a sequence of μ-simple functions.

Theorem: If $f(s)$ and $g(s)$ are measurable functions, then the following functions are also measurable:

1. $af(s)$, where $a \in \mathcal{R}$,
2. $|f(s)|$,
3. $f(s) + g(s)$
4. $f(s)g(s)$.

Proof: Items (1) and (2) follow immediately, since af and $|f|$ are continuous functions of f. Item (3) follows since f and g are the limits of sequences of μ-simple functions, and hence $f + g$ must be also. For item (4), consider:

$$f(s)g(s) = \frac{1}{4}\left\{[f(s) + g(s)]^2 - [f(s) - g(s)]^2\right\}. \tag{35}$$

But $-g$ is measurable by (1), and $f+g$ and $f-g$ are then measurable by (3). Finally the square of a function is continuous, hence measurable.

Definition (*Almost everywhere*): For given (S, Σ, μ), if a property P holds at all points of S except possibly a set of measure zero, P is said to hold **almost everywhere** in S.

Definition (*Equvalence*): Two functions $f(s)$ and $g(s)$ defined on S are called **equivalent** if $f(s) = g(s)$ almost everywhere. [Note that "defined on S" means assumes finite values at every point of S.] If f and g are equivalent functions, we write:

$$f \sim g. \tag{36}$$

Theorem: If $f \sim g$ and if g is measurable, then f is measurable. If a sequence of measurable functions converges almost everywhere to a function f, then f is measurable.

5 Integration

We are ready to generalize the Riemann notion of an integral. First, let us review the Riemann integral (see Fig.5). Let $f(x)$ be a continuous function on $a \leq x \leq b$. Then we define the Riemann integral of f as:

$$\begin{aligned} I_R &= \int_a^b f(x)\, dx && (37) \\ &= \lim_{\{\Delta_k\}\to 0} \sum_{k=1}^{N} f(x_k)\Delta_k, && (38) \end{aligned}$$

where we have divided the interval $a \leq x \leq b$ into N intervals of widths $\Delta_1, \Delta_2, \ldots, \Delta_N$, and x_k is a value of x in the kth interval.

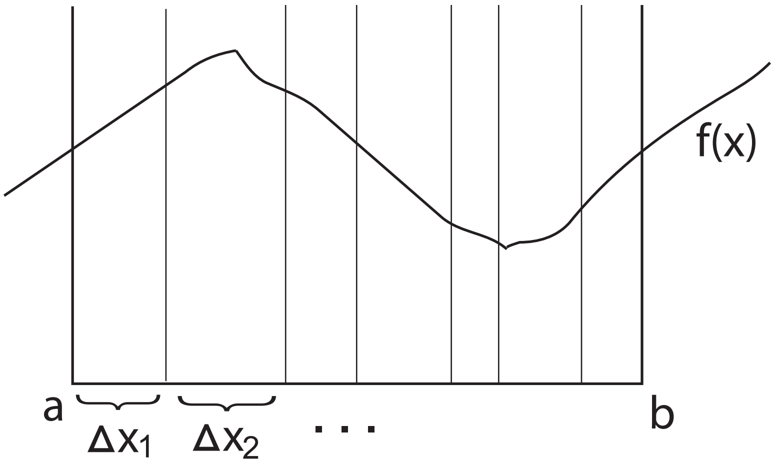

Figure 6: Illustration of the Riemann integral.

This definition of the integral depends on f changing only slightly within the intervals, in the limit where they are made small. It can still work, however, even if f has some discontinuities. It won't work for all functions, for example:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases} \tag{39}$$

The procedure for I_R becomes ill-defined in this case. More generally, it can be shown that a function f is Riemann-integrable on interval $[a, b]$ iff:

1. f is bounded;

2. the set of points of discontinuity of f has Lebesgue measure zero.

We may define an integral that works in situations where the Riemann doesn't by turning the picture we had for the Riemann integral "on its side" (see FIg.5). The result is the Lebesgue integral. The idea is to partition the $y = f$ axis instead of the x axis into intervals, Δy_i, $i = 1, 2, \ldots$ Choose a value y_i within each such interval. Then consider the sets $f^{-1}(\Delta y_i)$ corresponding to the inverse mapping of each interval. Multiply the measure of each such set by the corresponding y_i and sum over all intervals. The result, in the limit where all the $\Delta y_i \to 0$, is the Lebesgue integral. The measure of $f^{-1}(\Delta y_i)$ must exist, that is, f must be a measurable function.

For the function defined in Eq. 39, in the limit of infinitely small intervals, the relevant values of y are $y_1 = 1$ and $y_2 = 0$. The inverse mappings are:

$$
\begin{aligned}
f^{-1}(\Delta y_1) &= \text{irrational numbers in } [a, b] && (40) \\
f^{-1}(\Delta y_2) &= \text{rational numbers in } [a, b]. && (41)
\end{aligned}
$$

The Lebesgue integral is

$$
\begin{aligned}
I_L &= y_1 \mu \left[f^{-1}(\Delta y_1) \right] + y_2 \mu \left[f^{-1}(\Delta y_2) \right] \\
&= 1 \times \mu(\text{irrational numbers in } [a, b]) + 0 \times \mu(\text{rational numbers in } [a, b]) \\
&= b - a, && (42)
\end{aligned}
$$

since the measure of the interval $[a, b]$ is $b - a$, and this interval is the union of the rationals and irrationals in the interval, but the measure of the rationals is zero, hence the measure of the irrationals is $b - a$.

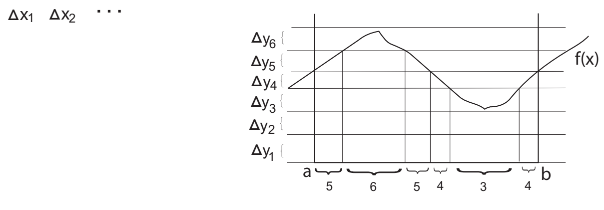

Figure 7: Illustration of the Lebesgue integral. The numbers below the horizontal axis indicate the sets consisting of the inverse image of f of the corresponding intervals along the vertical axis.

Let's make this definition a bit more formal. First, assume f is a measurable function and $A \subset S$ is a measurable set. We'll also assume for simplicity here that $\mu(S) < \infty$. Let

$$f(s) = \sum_{n=1}^{\infty} a_n \chi_{A_n}(s) \tag{43}$$

be a μ-simple function with distinct a_n's. Partition the real numbers into intervals Δy_n so that each interval includes exactly one a_n. In this case,

$$f^{-1}(\Delta y_n) = A_n, \tag{44}$$

and hence we define the integral as follows:
Definition (*Summable μ-simple function*): The μ-simple function f is called **summable** on a measurable set A if the series

$$\sum_{n=1}^{\infty} a_n \mu(A \cap A_n) \tag{45}$$

is absolutely convergent. In this case, the integral of f with respect to the measure μ is:

$$\int_A f(s)\mu(ds) \equiv \sum_{n=1}^{\infty} a_n \mu(A \cap A_n). \tag{46}$$

Definition (*Summable on A*): A measurable function f is called **summable on** A if it is the uniform limit of a sequence, $f_1, f_2, \ldots$, of summable μ-simple functions on A. In this case,

$$\int_A f(s)\mu(ds) \equiv \lim_{n\to\infty} \int_A f_n(s)\mu(ds). \tag{47}$$

It is readily seen that the Lebesque integral defines a linear functional on the vector space of summable functions. Some elementary properties include:

- For constants k_1, k_2,

$$\int_A \left[k_1 f(s) + k_2 g(s)\right] \mu(ds) = k_1 \int_A f(s)\mu(ds) + k_2 \int_A g(s)\mu(ds). \tag{48}$$

- If $|f(s)| \leq M$ for all $s \in A$, then

$$\left| \int_A f(s)\mu(ds) \right| \leq M\mu(A). \tag{49}$$

- For every summable function f,

$$\left| \int_A f(s)\mu(ds) \right| \leq \int_A |f(s)|\mu(ds). \tag{50}$$

The Lebesgue integral is a generalization of the Riemann integral, not only because it can handle functions with difficult discontinuities, but also because we don't even have to use the Lebesgue measure. For example, we could use the Dirac measure: Let x_0 be a real number. Define the measure

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A. \end{cases} \tag{51}$$

In this case, the semi-ring on which μ is defined includes all subsets of $\mathcal{R}$, $\mu^* = \mu$, and all subsets of $\mathcal{R}$ are measurable. Given a μ-simple function, $f(x) = \sum_{n=1}^{\infty} a_n \chi_{A_n}(x)$, the point x_0 must belong to one of the sets, say A_k. Then

$$\int_{\mathcal{R}} f(x)\mu(dx) = \sum_{n=1}^{\infty} a_n \mu(A_n) = a_k = f(x_0). \tag{52}$$

This measure achieves the effect of our δ-functional":

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0). \tag{53}$$

An important property of the Lebesgue integral is that if f_1 and f_2 are two equivalent summable functions, then:

$$\int_A f_1(s)\mu(ds) = \int_A f_2(s)\mu(ds). \tag{54}$$

Thus, if we decompose the set of summable functions into classes of equivalent functions, the integral may be regarded as a functional defined on the space of these classes. This idea is used in the construction of the Hilbert space, L^2, which we now turn to.

6 The Space L^2

We have been dealing with real functions so far, but the notion of the Lebesgue integral can be extended to complex functions: A complex function $f(s) = f_1(s) + i f_2(s)$, where f_1 and f_2 are real functions, is said to be summable on A if f_1 and f_2 are summable. In this case,

$$\int_A f(s)\mu(ds) = \int_A f_1(s)\mu(ds) + i \int_A f_2(s)\mu(ds). \tag{55}$$

Theorem: A complex function $f(s)$ is summable iff its absolute value is summable, where

$$|f(s)| = \sqrt{f_1(s)^2 + f_2(s)^2}. \tag{56}$$

Many other properties of the integral, such as linearity and denumerable additivity, carry over to this complex case.

Let L stand for the set of functions f such that $|f|^2$ is summable on S:

$$\int_S |f(s)|^2 \mu(ds) < \infty. \tag{57}$$

Theorem: L is a linear space.

Consider the subset $Z \subset L$ of functions f such that

$$\int_S |f(s)|^2 \mu(ds) = 0. \tag{58}$$

Note that Z is a linear subspace of L, since if $f \in Z$ then $kf \in Z$, where k is a constant, and if $f, g \in Z$ then $f + g \in Z$. The latter statement follows since $|f + g| \leq 2|f|^2 + 2|g|^2$.

Now define the "factor space":

$$L^2 = L/Z. \tag{59}$$

That is, two functions $f_1, f_2 \in L$ determine the same class in L^2 iff

$$\int_S |f1(s) - f_2(s)|^2 \mu(ds) = 0. \tag{60}$$

The difference $f_1 - f_2$ vanishes almost everywhere, $f_1 \sim f_2$. Hence, we say that the space L^2 consists of functions f such that $|f|^2$ is summable on S with the understanding that equivalent functions are not considered unique. In other words, L^2 is a space of equivalence classes.

Theorem: If $f, g \in L^2$ then $\bar{f}g$ is summable on S.

Proof:

$$\bar{f}g = \frac{1}{4}\left(|f+g|^2 - |f-g|^2 + i|f-ig|^2 - i|f+ig|^2\right) \tag{61}$$

Each term on the right side is summable, hence $\bar{f}g$ is summable.

Theorem: L^2 is a Hilbert space, with scalar product defined by:

$$\langle f|g\rangle \equiv \int_S \bar{f}(s)g(s)\mu(ds), \tag{62}$$

where μ is a given measure (e.g., Lebesgue, Dirac, etc.).

The proof must show first that L^2 is a pre-Hilbert space. This is relatively simple. Note that the requirement that $\langle f|f\rangle = 0$ iff $f = 0$ is satisfied because the equivalence class $f \sim 0$ is considered to be a unique "$f = 0$". The second part of the proof is to show that L^2 is complete. For this, it must be shown that every Cauchy sequence of vectors in L^2 converges to a vector in L^2.

It is a postulate of quantum mechanics that to every physical system S there corresponds a separable Hilbert space H_S.

7 Exercises

1. Show that, with suitable measure, any summation over discrete indices may be written as a Lebesgue integral:

$$\sum_{n=1}^{\infty} f(x_n) = \int f(x)\mu(dx). \tag{63}$$

2. Let $f \in L^2(-\pi, \pi)$ be a summable complex function on the real interval $[-\pi, \pi]$ (with Lebesgue measure).

   (a) With scalar product defined by:

$$\langle f|g\rangle \equiv \int_{-\pi}^{\pi} f^*(x)g(x)dx, \tag{64}$$

   for all $f, g \in L^2(-\pi, \pi)$, show that any f may be expanded as

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{65}$$

   where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx & (66)\\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \quad n > 0 & (67)\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx, \quad n > 0. & (68)\end{aligned}$$

   You may take as given that there is no vector in $L^2(-\pi, \pi)$ other than the trivial vector that is orthogonal to all of the functions $\sin nx$, $\cos nx$, $n = 0, 1, 2, \ldots$ What I have in mind here is that you need to show that the norm of the difference between the function and its above proposed expansion is zero, since in this case the expansion and the function are equal.

   (b) Now consider the function

$$f(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0. \end{cases} \tag{69}$$

   What is the expansion of this function in terms of the expansion of part (a)?

(c) We wish to examine the partial sums in the expansion of part (b):

$$f_N(x) \equiv \sum_{n=0}^{N} \left(a_n \cos nx + b_n \sin nx\right). \tag{70}$$

Find the position, x_N, of the first maximum of f_N (for $x_N > 0$). Evaluate the limit of $f_N(x_N)$ as $N \to \infty$. Give a numerical answer, accurate to 1%, say. In doing this, you are finding the maximum value of the series expansion in the limit of an infinite number of terms. You may, if you wish, use the following fact:

$$\mathrm{Si}(\pi) \equiv \int_0^{\pi} \frac{\sin x}{x} dx \approx 1.8519. \tag{71}$$

(d) The maximum value of $f(x)$, as defined in part (b), is one. If the value you found for the series expasnion in part (c) is different from one, comment on the possible reconciliation of this with the theorem you demonstrated in part (a).