

Physics 125c
Problem set number 1 – Solution to Problem 1
Due Wednesday, April 7, 2004

PROBLEMS:

1. Let us try an example of the discussion we have had in class concerning the use of the uncertainty relation on “localized” wave functions. Consider the three dimensional generalization. Hence, let $P(a)$ be the probability to find a particle of mass m in a sphere of radius a centered at the origin.
 - (a) Recall that in the one dimensional case, if the probability of finding the particle in the interval $(-a, a)$ was α , then a simple lower bound on the kinetic energy was obtained as:

$$T \geq \frac{1}{8m} \frac{\alpha^2}{a^2}. \quad (1)$$

Make a simple, but rigorous, generalization of this result to the three dimensional case. Don't worry about finding the “best” bound; even a “conservative” bound may be good enough to answer some questions of interest.

Solution: The limit on T in Eqn. 1 corresponds to a limit on the momentum of

$$\langle p^2 \rangle \geq \frac{1}{4} \frac{\alpha^2}{a^2}. \quad (2)$$

In three dimensions the kinetic energy is

$$T = \frac{1}{2m} \langle p_x^2 + p_y^2 + p_z^2 \rangle. \quad (3)$$

If the probability to find the particle within radius a is $P(a)$, then in each dimension we certainly have that the probability to be in the interval $(-a, a)$ is at least $P(a)$. Thus, we can apply Eqn. 2 in each dimension, and hence, in sum:

$$T \geq \frac{3}{8m} \frac{P(a)^2}{a^2}. \quad (4)$$

- (b) We know that an atomic size is of order 10^{-10} m. Suppose that we have an electron which is known to be in a sphere of radius 10^{-10} m with 50% probability. What lower bound can you put on its kinetic energy? Is the result consistent with expectation; *e.g.*, with what you know about the kinetic energy of the electron in hydrogen?

Solution:

$$\begin{aligned} T &\geq \frac{3}{8} \left(\frac{1}{2}\right)^2 \frac{(200 \text{ MeV}\cdot\text{fm})^2}{0.5 \text{ MeV} 10^{-20} \text{ m}^2} \\ &\geq 0.8 \text{ eV}. \end{aligned} \tag{5}$$

In the ground state, the expectation value of the kinetic energy of the electron in hydrogen is 13.6 eV, consistent with our bound, although our bound is not especially good.

- (c) In ancient times, before the neutron was discovered, it was supposed that the nucleus contained both electrons and protons. A comfortable nuclear size is 5×10^{-15} m. Find a lower bound on the kinetic energy of an electron if the probability to be within this radius is 90%. If there is a problem with the validity of your bound, see if you can fix it. Does this model encounter any difficulties?

Solution:

$$\begin{aligned} T &\geq \frac{3}{8} (0.9)^2 \frac{(200 \text{ MeV}\cdot\text{fm})^2}{0.5 \text{ MeV} 25 \times 10^{-30} \text{ m}^2} \\ &\gtrsim 1 \text{ GeV}. \end{aligned} \tag{6}$$

The electron is relativistic, inconsistent with our assumption in computing the kinetic energy with a non-relativistic equation. Our derivation that $\langle p^2 \rangle \geq 1/4a^2$ in the one-dimensional case should still be valid. The relativistic kinetic energy is $T = E - m \approx \langle |\mathbf{p}| \rangle$, where the approximation is in the limit $E \gg m$. Let's presume we can estimate T with $\sqrt{\langle p^2 \rangle}$. Then

$$\begin{aligned} T &\geq \frac{\sqrt{3}}{2a} \\ &\geq \frac{200 \text{ MeV}\cdot\text{fm}}{5 \text{ fm}} = 40 \text{ MeV}. \end{aligned} \tag{7}$$

Let's compare this with an estimated order of magnitude for the electrostatic potential energy of an electron and a proton separated by 5 fm:

$$|V| = \frac{e^2}{a} \approx \frac{200 \text{ MeV}\cdot\text{fm}}{100 \times 5 \text{ fm}} = 0.4 \text{ MeV} \quad (8)$$

This is much smaller than our limit on the kinetic energy, presenting a theoretical problem with binding and with expectations from the virial theorem.

- (d) Now find a lower bound for the kinetic energy of a proton in the nucleus, if it has a probability of 90% to be within a region of radius 5×10^{-15} m.

Solution:

$$\begin{aligned} T &\geq \frac{3}{8} (0.9)^2 \frac{(200 \text{ MeV}\cdot\text{fm})^2}{900 \text{ MeV} 25 \times 10^{-30} \text{ m}^2} \\ &\geq 0.5 \text{ MeV}. \end{aligned} \quad (9)$$

2. Do exercise 1 of the "Approximate Methods" course note.
3. Do exercise 2 of the "Approximate Methods" course note.
4. Do exercise 3 of the "Approximate Methods" course note.