

Physics 125c
 Problem set number 8 – Solution to Problem 28
 Due Wednesday, May 26, 2004

PROBLEMS:

28. Let us pursue further the application of our discussion on electromagnetic interactions in atomic physics. Consider in particular the common case in which the wavelength of light emitted by an atom is much larger than the atomic dimensions. Recall that the transition rate for spontaneous emission from atomic state $|n\rangle$ to state $|0\rangle$ is:

$$\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{sp em}; n \rightarrow 0) = \frac{4\pi^2 q^2}{\omega V} \delta(E_n - E_0 - \omega) |\langle 0 | \hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^* | n \rangle|^2. \quad (29)$$

- (a) Show that, to a good approximation we may evaluate the atomic transition matrix element according to:

$$\langle 0 | \hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^* | n \rangle = -i\omega \langle 0 | \mathbf{x} | n \rangle \cdot \boldsymbol{\epsilon}^*, \quad (30)$$

where $\mathbf{x} = \sum_i \mathbf{x}_i$, which may be referred to as the dipole moment operator (multiply it times q , and you have the electric dipole operator). This approximation is known as the electric dipole approximation. [Hint: Evaluate the commutator of \mathbf{x} with H_0 .]

Solution: Start with:

$$\langle 0 | \hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^* | n \rangle = \int d^3(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \langle 0 | \mathbf{j}(\mathbf{x}) | n \rangle \cdot \boldsymbol{\epsilon}^*. \quad (31)$$

The long wavelength limit corresponds to k being very small, that is $\mathbf{k} \cdot \mathbf{x} \ll 1$. Then:

$$\langle 0 | \hat{\mathbf{j}}(\mathbf{k}) | n \rangle = \langle 0 | \int d^3(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{j}(\mathbf{x}) | n \rangle, \quad (32)$$

$$\approx \langle 0 | \int d^3(\mathbf{x}) \mathbf{j}(\mathbf{x}) | n \rangle \quad (33)$$

$$= \langle 0 | \frac{1}{2m} \sum_i \int d^3(\mathbf{x}) [\mathbf{p}_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \mathbf{p}_i] | n \rangle$$

$$= \frac{1}{m} \langle 0 | \mathbf{p} | n \rangle, \quad (34)$$

where

$$\mathbf{p} \equiv \sum_i \mathbf{p}_i. \quad (35)$$

But $[\mathbf{x}, H_0] = [\mathbf{x}, \mathbf{p}^2/2m] = i\mathbf{p}/m$. Hence,

$$\langle 0|\hat{\mathbf{j}}(\mathbf{k})|n\rangle = -i\langle 0|[\mathbf{x}, H_0]|n\rangle \quad (36)$$

$$= -i(E_n - E_0)\langle 0|\mathbf{x}|n\rangle \quad (37)$$

$$= -i\omega\langle 0|\mathbf{x}|n\rangle. \quad (38)$$

- (b) Using your result, estimate the lifetime of the 1P_1 state of atomic hydrogen (in the lowest radial state), as it decays to the 1S_0 ground state.

Solution: The lifetime τ will be the inverse of the total decay rate: $\tau = 1/\Gamma$. We evaluate Γ :

$$\Gamma = \sum_{\mathbf{k}\boldsymbol{\epsilon}} \Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{sp em}; n \rightarrow 0) \quad (39)$$

$$= \frac{4\pi^2 e^2}{V} \sum_{\mathbf{k}\boldsymbol{\epsilon}} \frac{1}{\omega} \delta(E_n - E_0 - \omega) |\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2 \quad (40)$$

$$= \frac{4\pi^2 e^2}{(2\pi)^3} \sum_{\boldsymbol{\epsilon}} \int_0^\infty \omega d\omega \delta(E_n - E_0 - \omega) \int_{(4\pi)} d\Omega_k |\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2$$

$$= \frac{4\pi^2 e^2}{(2\pi)^3} \sum_{\boldsymbol{\epsilon}} \omega \int_{(4\pi)} d\Omega_k |\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2 \quad (41)$$

$$= \frac{e^2 \omega^3}{2\pi} \sum_{\boldsymbol{\epsilon}} \int_{(4\pi)} d\Omega_k |\langle 0|\mathbf{x}|n\rangle \cdot \boldsymbol{\epsilon}^*|^2 \quad (42)$$

Now, the hydrogen wave functions may be expressed in the form:

$$\psi_{n\ell m}(\mathbf{x}) = R_{n\ell}(r)Y_{\ell m}(\theta, \phi), \quad (43)$$

where $r \equiv |\mathbf{x}|$ and (θ, ϕ) are respectively the polar and azimuth angles of \mathbf{x} in spherical coordinates. The label n is the principal (or total) quantum number, and ℓ is the orbital angular momentum quantum number. For our present case, $n = 1$ or 2 , and ℓ is 0 or 1 , for the S and P states, respectively. The specific wave functions

we want are:

$$\psi_{100}(\mathbf{x}) = 2 \left(\frac{1}{a_0} \right)^{(3/2)} e^{-r/a_0} Y_{00}(\theta, \phi) \quad (44)$$

$$\psi_{21m}(\mathbf{x}) = \left(\frac{1}{2a_0} \right)^{(3/2)} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0} Y_{1m}(\theta, \phi), \quad (45)$$

where $a_0 = 1/m_e\alpha$ is the Bohr radius, and

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \quad (46)$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta \quad (47)$$

$$Y_{1\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \quad (48)$$

We'll evaluate things by taking the radiated photon to define the z axis. The polarization of the initial hydrogen atom with respect to the photon direction is unspecified: we must average over the three possible polarizations. The matrix elements to evaluate are:

$$\langle 100|\mathbf{x}|21m \rangle = \frac{a_0}{\sqrt{6}} \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{r^4 dr}{a_0^5} e^{-\frac{3r}{2a_0}} \quad (49)$$

$$\int_{-1}^1 d \cos \theta \int_0^{2\pi} d\phi Y_{1m}(\theta, \phi) (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$= a_0 \sqrt{\frac{24}{\pi}} \left(\frac{2}{3} \right)^5 \quad (50)$$

$$\int_{-1}^1 d \cos \theta \int_0^{2\pi} d\phi Y_{1m}(\theta, \phi) (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

Notice that if \mathbf{k} is along the z axis, the z component of the photon polarization is zero. As $\cos \theta \sin \theta$ is an odd function of $\cos \theta$, this means that only the $m = \pm 1$ components will contribute. Thus,

$$\langle 100|\mathbf{x}|21 \pm 1 \rangle = a_0 \sqrt{\frac{24}{\pi}} \left(\frac{2}{3} \right)^5 \left(\mp \sqrt{\frac{3}{8\pi}} \right) \int_0^{2\pi} d\phi e^{\pm i\phi} \quad (51)$$

$$\int_{-1}^1 d \cos \theta \sin^2 \theta (\cos \phi, \sin \phi, 0) \quad (52)$$

$$\begin{aligned}
&= \mp a_0 \frac{3}{\pi} \left(\frac{2}{3}\right)^5 \int_0^{2\pi} d\phi e^{\pm i\phi} \frac{4}{3} (\cos \phi, \sin \phi, 0) \quad (53) \\
&= a_0 4 \left(\frac{2}{3}\right)^5 (\mp 1, -i, 0). \quad (54)
\end{aligned}$$

We now go back to our equation for the rate, and consider the sum over polarizations. With \mathbf{k} in the z direction, we have polarizations in the $x-y$ plane. We may use $\boldsymbol{\epsilon} = (1, 0, 0)$ and $\boldsymbol{\epsilon} = (0, 1, 0)$. Then,

$$\sum_{\boldsymbol{\epsilon}} |\langle 0 | \mathbf{x} | n \rangle \cdot \boldsymbol{\epsilon}^*|^2 = a_0^2 16 \left(\frac{2}{3}\right)^{10} 2, \quad (55)$$

and hence,

$$\Gamma = \frac{e^2 \omega^3}{2\pi} a_0^2 32 \left(\frac{2}{3}\right)^{10} 4\pi \quad (56)$$

$$= \frac{2^{16}}{3^{10}} \frac{\omega^3}{\alpha m_e^2}. \quad (57)$$

The energy ω , neglecting nuclear recoil, is the difference in the atomic energy levels. The energy levels are given by:

$$E_n = -\frac{m_e \alpha^2}{2} \frac{1}{n^2} = \frac{-13.6}{n^2} \text{ eV}. \quad (58)$$

Thus, $\omega = 3\alpha^2 m_e / 8 = 10.2 \text{ eV}$, and

$$\Gamma = \left(\frac{2}{3}\right)^7 \alpha^5 m_e \quad (59)$$

$$= 6.2 \times 10^{-7} \text{ eV} \quad (60)$$

$$= 9.4 \times 10^8 \text{ s}^{-1}. \quad (61)$$

Thus, the lifetime is $\tau = 1/\Gamma = 1.1 \times 10^{-9} \text{ s}$.

29. Do exercise 1 of the “Density Matrix Formalism” course note.
30. Do exercise 2 of the “Density Matrix Formalism” course note.
31. Do exercise 3 of the “Density Matrix Formalism” course note.