Physics 125c
Problem set number 10 – Solution to Problem 38
Due Wednesday, June 9, 2004

PROBLEMS:

36. Proof of the von Neumann mixing theorem: Do exercise 4 of the "Density Matrix Formalism" course note.

37. Do exercise 7 of the "Density Matrix Formalism" course note.

38. Electric dipole selection rules: In problem 28 we investigated the life time of a hydrogen level under the long wavelength "electric dipole" approximation. It is conceivable the the result of such a computation could be zero, in some cases, that is, the transition is forbidden in this approximation (but perhaps allowed in higher orders in the expansion of the $e^{i\mathbf{k}\cdot\mathbf{x}}$ expansion). Consider transitions between states of specified initial and final orbital angular momentum, ℓ_i and ℓ_f, and initial and final projections of the orbital angular momentum on the z-axis, m_i and m_f. In the electric dipole approximation, what transitions are permitted (e.g., what are the permitted values of ℓ_f and m_f if ℓ_i and m_i are given)?

    Solution: In the electric dipole approximation, we are interested in matrix elements of the position operator:

$$\begin{aligned}\langle n_f\ell_f m_f|\mathbf{x}|n_i\ell_i m_i\rangle \quad \propto \quad & \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \\ & Y^*_{\ell_f m_f}(\cos\theta,\phi)(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)Y_{\ell_i m_i}(\cos\theta,\phi).\end{aligned} \tag{62}$$

    We can do this problem without working very hard: First, we note that

$$Y^*_{\ell_f m_f}(\cos\theta,\phi) = (-)^{m_f} Y_{\ell_f -m_f}(\cos\theta,\phi). \tag{63}$$

    As a consequence of the Clebsch-Gordan series, we know that the product of spherical harmonics $Y_{\ell_f -m_f}(\cos\theta,\phi)Y_{\ell_i m_i}(\cos\theta,\phi)$ may be written as a sum over spherical harmonics $Y_{\ell m}(\cos\theta,\phi)$, where ℓ ranges from $|\ell_f-\ell_i|$ to $\ell_f+\ell_i$, and all ℓ terms are present. Furthermore, only terms with $m = m_i - m_f$ appear in the sum. The desired matrix element thus involves terms of the form

$$\int_{-1}^{1} d\cos\theta \int_{0}^{2\pi} Y_{\ell m_i - m_f}(\cos\theta, \phi)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$
$$\ell = |\ell_f - \ell_i|, \ldots, \ell_f + \ell_i. \tag{64}$$

Now recall that:

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \tag{65}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{66}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \tag{67}$$

Hence,

$$\sin\theta\cos\phi = \sqrt{\frac{2\pi}{3}}(-Y_{11}^* + Y_{1-1}^*) \tag{68}$$

$$\sin\theta\sin\phi = -i\sqrt{\frac{2\pi}{3}}(Y_{11}^* + Y_{1-1}^*) \tag{69}$$

$$\cos\theta = \sqrt{\frac{4\pi}{3}}Y_{10}^* \tag{70}$$

Finally, we use the orthogonality of the spherical harmonics to deduce that the matrix element will be zero unless the set $\ell = |\ell_f - \ell_i|, \ldots, \ell_f + \ell_i$ contains the value one, and $m_i - m_f = \pm 1$ or 0. That is, our selection rules are:

$$\Delta\ell \equiv \ell_f - \ell_i = \pm 1 \text{ or } 0, \text{ except } \ell_f = \ell_i = 0 \text{ is forbidden,} \tag{71}$$

and

$$\Delta m \equiv m_f - m_i = \pm 1 \text{ or } 0. \tag{72}$$