Ph 129B: (Easier) Proof of the Invariance of the Translation Subgroup in the Poincaré Group

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Let $\varphi : \Lambda(M, z) \mapsto \Lambda(M, 0)$. We can easily verify that φ is a homomorphism of $\overline{L}$.

Consider the set $\{\Lambda : \varphi(\Lambda) = I = \Lambda(I, 0)\}$. We can see that this is exactly the subgroup of translations: no other members of $\overline{L}$ have an identity Lorentz component, and all translations are sent to $I = \Lambda(I, 0)$ since our homomorphism throws away the translation parameter z. We will use this "alternate definition" of Tr shortly: $t \in \mathrm{Tr}$ iff $\varphi(t) = I$.

Let $t \in \mathrm{Tr}$ and $g \in \overline{L}$. Using that $\varphi(t) = I$, we get

$$\begin{aligned}\varphi(gtg^{-1}) &= \varphi(g)\varphi(t)\varphi(g)^{-1} \\ &= \varphi(g)I\varphi(g)^{-1} \\ &= I.\end{aligned}$$

Using the other direction of our iff, this means that $gtg^{-1} \in \mathrm{Tr}$. Since t and g were arbitrary, this means T satisfies our definition of an invariant subgroup. $\square$

We can see that this is actually a very general method for proving invariance of a subgroup. The theorem is that a subgroup is invariant iff there is some homomorphism sending it to the identity. (In group-theoretical language, "a normal subgroup is the kernel of a homomorphism.") This also isn't tied to any particular representation, which is nice.