# ph129 PS 2 solutions

Sean Tulin (revised by C. Park)

October 23, 2007

## Problem 5(C. Park)

The Fourier transform is given by

$$f(\mathbf{y}) = \frac{1}{2\pi}\int d^2x \frac{e^{-\mu r}}{r} e^{i\mathbf{x}\cdot\mathbf{y}}$$

where $r = |y|$. In the polar coordinate system $(r, \theta)$ for the $\mathbf{x}$ vector,

$$\begin{aligned} f(\mathbf{y}) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} drr\frac{e^{-\mu r}}{r} e^{iry\cos\theta} && (1)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} dre^{-\mu r + iry\cos\theta} && (2)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{\mu - iy\cos\theta} && (3)\end{aligned}$$

Integration over $r$ makes sense since the real part of the exponent is negative($\mu > 0$). To integrate over $\theta$ introduce a complex variable $z = e^{i\theta}$ and treat the equation above as a contour integral over the unit circle in the complex plane;

$$\begin{aligned} f(\mathbf{y}) &= \frac{1}{2\pi}\oint \frac{dz}{iz}\frac{1}{\mu - \frac{iy}{2}\left(z + \frac{1}{z}\right)} && (4)\\ &= \frac{1}{2\pi}\frac{2}{y}\oint dz \frac{1}{z^2 - \frac{2}{iy}\mu z + 1} && (5)\end{aligned}$$

Notice that the product of the two roots of the denominator in the integrand is 1. That means one of the roots lies inside the unit circle and the other outside. Explicitly,

$$z = -\frac{i\mu}{y} + i\sqrt{\frac{\mu^2}{y^2} + 1}$$

is in the unit circle. Using the residue theorem,

$$\begin{aligned} f(\mathbf{y}) &= 2\pi i \frac{1}{2\pi}\frac{2}{y}\frac{1}{2i\sqrt{1 + \frac{\mu^2}{y^2}}} && (6)\\ &= \frac{1}{\sqrt{\mu^2 + y^2}} && (7)\end{aligned}$$

## Problem 6

### N=1

We want to solve this equation

$$f(x) = x + \int_0^x dy\, e^{-xy}\, f(y), \tag{8}$$

which means $K(x,y) = e^{-xy}$ and $g(x) = x$. We want to evaluate $f(1)$, and so the integral on the RHS goes from 0 to 1. The first task is to divide this integral into $N+1$ discrete steps. For $N = 1$, we just have two values: $x_0 = 0$ and $x_1 = 1$, with $\Delta = 1$. Next, we evaluate $g$ and $K$ at these discrete points. If we denote $g_i = g(x_i)$ and $K_{ij} = K(x_i, x_j)$, then we have

$$g_i = (0, 1) \qquad\qquad K_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & e^{-1} \end{pmatrix} . \tag{9}$$

Now we can start with our iterative solution. First, $f_0 = g_0 = 0$. Then, the next iteration gives

$$\left(1 - \frac{1}{2}\, K_{11}\right)\, f_1 = g_1 + \left(\frac{1}{2}\, K_{10} f_0\right) . \tag{10}$$

Plugging in and solving for $f_1$, we find

$$f(1) = f_1 = \frac{1}{1 - 1/2e} = 1.225 . \tag{11}$$

### N=2

Now we have three discrete points: $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$, with $\Delta = 1/2$. Evaluating $K$ and $g$ at these points, we find

$$g_i = \left(0, \frac{1}{2}, 1\right) \qquad\qquad K_{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-1/4} & e^{-1/2} \\ 1 & e^{-1/2} & e^{-1} \end{pmatrix} . \tag{12}$$

The first iteration is the same: $f_0 = g_0 = 0$. For the next iteration we find

$$\left(1 - \frac{1}{4}\, K_{11}\right)\, f_1 = g_1 + \left(\frac{1}{4}\, K_{10} f_0\right) . \tag{13}$$

which gives $f_1 = 0.621$. In the final iteration, we find

$$\left(1 - \frac{1}{4}\, K_{22}\right)\, f_2 = g_2 + \frac{1}{2}\left(\frac{1}{2}\, K_{20} f_0 + K_{21} f_1\right) . \tag{14}$$

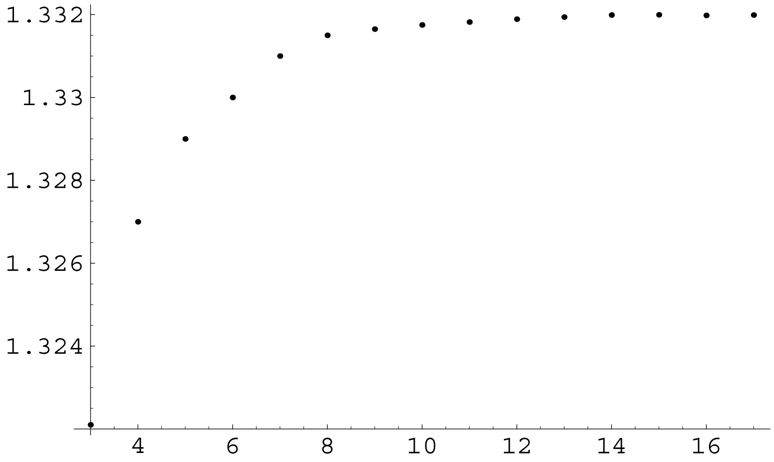
which gives $f_2 = f(1) = 1.309$.

### N=3

Now $\Delta = 1/3$, so we have four discrete points: $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$. For $K$ and $g$, we have

$$g_i = \left(0, \frac{1}{3}, \frac{2}{3}, 1\right) \qquad K_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-1/9} & e^{-2/9} & e^{-1/3} \\ 1 & e^{-2/9} & e^{-4/9} & e^{-2/3} \\ 1 & e^{-1/3} & e^{-2/3} & e{-1} \end{pmatrix} . \tag{15}$$

After doing the same iterative procedure as above, we find $f_1 = 0.391$, $f_2 = 0.864$, and finally $f(1) = f_3 = 1.322$. So we can see that $f(1)$ really does converge by increasing $N$. I've used Mathematica to repeat the procedure for larger $N$ and the following plot shows $f(1)$ as a function of $N$.



## Problem 7

The Laguerre equation

$$xf'' + (1-x)f' + \lambda f = 0 \tag{16}$$

is an equation of the form

$$\sum_{k=0}^{n} (a_k + b_k x) f^{(k)}(x) = 0 \tag{17}$$

with

$$a_2 = 0 \qquad b_2 = 1 \tag{18}$$

$$a_1 = 1 \qquad b_1 = -1 \tag{19}$$

$$a_2 = \lambda \qquad b_0 = 0\ . \tag{20}$$

Thus we have

$$
\begin{aligned}
u(s) &= \sum_{k=0}^{n} a_k s^k = s + \lambda && (21)\\
v(s) &= \sum_{k=0}^{n} b_k s^k = s(s-1)\ . && (22)
\end{aligned}
$$

And then we get

$$
\begin{aligned}
F(s) &= \frac{A}{v(s)} \exp\left(\int^s ds' \, \frac{u(s')}{v(s')}\right) && (23)\\
&= \frac{A}{s(s-1)} \exp\left(\int^s ds' \, \frac{\lambda + s'}{s'(s'-1)}\right) && (24)\\
&= \frac{A}{s}\left(\frac{s-1}{s}\right)^{\lambda}\ . && (25)
\end{aligned}
$$

The solution to the Laguerre equation is now simply

$$f(x) = \int_C ds\, F(s)\, e^{sx} \tag{26}$$

where our contour $C$ is any contour that satifies the following condition:

$$v(s)\, F(s)\, e^{sx}|_{C_1}^{C_2} = 0\ . \tag{27}$$

If we let $\lambda = n = 0, 1, 2...$, then there are no branch cuts in $F(s)$. There is only a pole of order $n+1$ at $s = 0$. Thus we can choose $C$ to be the unit circle around the origin, which trivially satifies the above condition, since $C_1 = C_2$. So now we have

$$
\begin{aligned}
f(x) &= 2\pi i \times Residue[F(s)e^{sx}; s = 0] && (28)\\
&= \frac{2\pi i}{n!} \lim_{s\to 0} \frac{d^n}{ds^n}\ (s^n\, F(s)\, e^{sx}) && (29)\\
&= \frac{A}{n!} \lim_{s\to 0} \frac{d^n}{ds^n}\ ((s-1)^n\, e^{sx}) && (30)
\end{aligned}
$$

where in the last line I've absorbed the factor of $2\pi i$ into the arbitrary constant $A$. We can write this in a slightly neater form through some algebra. First, if we let $t = s - 1$, then we have

$$\begin{aligned}
f(x) &= \frac{A}{n!}\lim_{t\to -1} e^x \frac{d^n}{dt^n}\left(t^n\, e^{tx}\right) && (31)\\
&= \frac{A}{n!}\lim_{t\to -1} e^x \sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)\left(\left(\frac{d}{dt}\right)^m e^{tx}\right)\left(\left(\frac{d}{dt}\right)^{n-m} t^n\right) && (32)\\
&= \frac{A}{n!}\lim_{t\to -1} e^x \sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) x^m e^{tx}\,\frac{n!}{m!} t^m && (33)\\
&= \frac{A}{n!} e^x \sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)(-x)^m e^{-x}\,\frac{n!}{m!} && (34)\\
&= \frac{A(-1)^n}{n!} e^x \frac{d^n}{dx^n}\left(x^n\, e^{-x}\right) && (35)
\end{aligned}$$

where

$$\left(\begin{array}{c} n \\ m \end{array}\right) = \frac{n!}{m!(n-m)!} \tag{36}$$

is the binomial expansion coefficient. Isolating part of this expression,

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(x^n\, e^{-x}\right) \tag{37}$$

is called Rodrigues' formula and $L_n$ is called a Laguerre polynomial.

## Problem 8

### A

We want to solve the equation

$$u(x) = e^x + \lambda \int_0^1 dt\; xt\; u(t)\ . \tag{38}$$

So we have $g(x) = e^x$ and degenerate kernel

$$K(x,y) = \phi(x)\psi(y) = xy \tag{39}$$

where $\phi(x) = x$ and $\psi(y) = y$. Using the method of degenerate kernels, we want to define

$$\begin{aligned} G &= \int_0^1 dx\, \psi(x) g(x) = \int_0^1 dx\, e^x x = 1 && (40) \\ C &= \int_0^1 dx\, \psi(x) \phi(x) = \int_0^1 dx\, x^2 = \frac{1}{3} && (41) \\ U &= \int_0^1 dx\, \psi(x) u(x)\ . && (42) \end{aligned}$$

The integral equation we want to solve is now

$$U = G + \lambda C U\ . \tag{43}$$

This gives

$$U = \frac{G}{1 - \lambda C} = \frac{3}{3 - \lambda} \tag{44}$$

and so

$$u(x) = g(x) + \lambda \phi(x) U = e^x + \frac{3\lambda x}{3 - \lambda}\ . \tag{45}$$

**B**

The kernel is degenerate and can be written as

$$K(x,t) = \sin(x - t) = \sin(x)\cos(t) - \cos(x)\sin(t) = \sum_{i=1,2} \phi_i(x)\psi_i(t) \tag{46}$$

where

$$\begin{aligned} \psi_1(x) &= \cos(x) & \psi_2(x) &= -\sin(x) && (47) \\ \phi_1(x) &= \sin(x) & \phi_2(x) &= \cos(x) && (48) \end{aligned}$$

Now we want to define

$$c_{ij} = \int_0^\pi dx\, \psi_i(x)\phi_j(x) = \begin{pmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{pmatrix}\ . \tag{49}$$

The integral equation is now

$$u_i = \lambda c_{ij} u_j \tag{50}$$

which implies

$$u_1 = \lambda \frac{\pi}{2} u_2 \qquad\qquad u_2 = -\lambda \frac{\pi}{2} u_1. \tag{51}$$

Solving these equations, we find that we must have $\lambda = \pm 2i/\pi$. These are the eigenvalues of our homogeneous integral equation. If we plug in the eigenvalues, we find that the eigenvectors are

$$u_i = A \begin{pmatrix} \pm i \\ 1 \end{pmatrix} \tag{52}$$

where $A$ is an arbitrary constant. Thus we have

$$u(x) = \sum_{i=1,2} u_i \phi(x) = A e^{\pm ix} \tag{53}$$

where $\pm$ in $u(x)$ corresponds to the particular eigenvalue $\lambda = \pm 2i/\pi$. If $\lambda \neq \pm 2i/\pi$, then we can have only the trivial solution $u(x) = 0$. Note that it is against the law to form linear combinations of the two solutions, e.g. if you wanted to write $u(x)$ in terms of sine and cosine. Each $u(x)$ is the solution of a different integral equation, because $\lambda$ is different in each case, so it doesn't make sense to combine them.

## Problem 9

We want to solve the equation

$$f(x) = x + \lambda \int_0^1 dy\, y(x+y)\, f(y)\ . \tag{54}$$

The kernel is $K(x,y) = y(x+y)$ and the inhomogenous piece is $g(x) = x$.

### A. Fredholm Method

The method goes as follows: First calculate the functions $D(\lambda)$ and $D(x,y;\lambda)$. Then the solution to the equation is simply

$$f(x) = g(x) + \int_0^1 dy\, g(y)\, \frac{D(x,y;\lambda)}{D(\lambda)}\ . \tag{55}$$

We want to work consistently to order $\lambda^2$, so we'll neglect all terms which are higher order. Then we have

$$\begin{aligned} D(x,y;\lambda) &= \lambda K(x,y) - \lambda^2 \int_0^1 dz (K(x,y)K(z,z) - K(x,z)K(z,y)) && (56) \\ &= \lambda y(x+y) - \lambda^2 \int_0^1 dz (y(x+y)2z^2 - z(x+z)y(z+y)) && (57) \\ &= \lambda y(x+y) - \frac{\lambda^2}{12}(4xy - 6y^2 + 4y^2 - 3y) && (58) \end{aligned}$$

and

$$\begin{aligned} D(\lambda) &= 1-\lambda \int_0^1 dx\, K(x,x) && (59)\\ &\quad +\frac{\lambda^2}{2!}\int_0^1 dx \int_0^1 dy\, (K(x,x)K(y,y)-K(x,y)K(y,x)) && (60)\\ &= 1-\frac{2\lambda}{3}-\frac{\lambda^2}{72}\ . && (61) \end{aligned}$$

We put the pieces together and find

$$\begin{aligned} f(x) &= x+\int_0^1 dy\, y\, \frac{D(x,y;\lambda)}{D(\lambda)} && (62)\\ &= x+\frac{\lambda(\frac{x}{3}+\frac{1}{4}+\frac{\lambda x}{72})}{1-\frac{2\lambda}{3}-\frac{\lambda^2}{72}}\ . && (63) \end{aligned}$$

In the limit of small $\lambda$, we can expand the denominator in powers of $\lambda$. Then we find

$$f(x)=x+\lambda(\frac{x}{3}+\frac{1}{4})+\lambda^2(\frac{17x}{72}+\frac{1}{6}) \tag{64}$$

## B. Neumann solution

The Neumann solution is an iterative solution. Let $f_n(x)$ denote $f(x)$ to order $\lambda^n$. We start with $f_0(x)=g(x)=x$. Then, by an iterative procedure, we can get $f_n(x)$ by

$$f_n(x)=x+\lambda\int_0^1 dy\, y(x+y)f_{n-1}(x)\ . \tag{65}$$

So we get

$$f_1(x)=x+\lambda\int_0^1 dy\, y(x+y)y=x+\lambda\left(\frac{x}{3}+\frac{1}{4}\right) \tag{66}$$

and

$$f_2(x)=x+\lambda\int_0^1 dy\, y(x+y)f_1(y)\ . \tag{67}$$

When we evaluate this we find that, to order $\lambda^2$,

$$f(x)=f_2(x)=x+\lambda(\frac{x}{3}+\frac{1}{4})+\lambda^2(\frac{17x}{72}+\frac{1}{6}) \tag{68}$$

which agrees with part A.