Ph129 PS 4 solutions

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Problem 15

We will use a method similar to what we used for degenerate kernels. Expanding $\sin k(s-t)$ in the equation,

$$\frac{d^2u}{dt^2}(t) + A\cos(kt) - B\sin(kt) = a(t) \tag{1}$$

where

$$A = \int_0^1 u(s)\sin(ks)ds, \qquad B = \int_0^1 u(s)\cos(ks)ds\ . \tag{2}$$

Integrating (1) twice,

$$u(t) = \frac{A}{k^2}\cos(kt) - \frac{B}{k^2}\sin(kt) + c(t) \tag{3}$$

where

$$c(t) = \iint u(s)ds\ .$$

For an indefinite integration, we need to specify one condition to fix an arbitrary constant. Since $c(t)$ is given by double integration, two conditions should be imposed on $c(t)$. Using $u(0) = 0$ and $u'(0) = 0$,

$$\begin{aligned} c(0) &= -\frac{A}{k^2} \\ c'(0) &= \frac{B}{k} \end{aligned} \tag{4}$$

These two conditions fix $c(t)$ completely once we have the values of A and B.

Using the expression for $u(t)$ in (3), (2) says

$$\begin{aligned} A &= \frac{A}{k}\int_0^1 dt\ \sin^2(kt) + \frac{B}{k}\int_0^1 dt\ \sin(kt)\cos(kt) + \int_0^1 dt\ c(t)\sin(kt) \\ B &= \frac{A}{k}\int_0^1 dt\ \sin(kt)\cos(kt) + \frac{B}{k}\int_0^1 dt\ \cos^2(kt) + \int_0^1 dt\ c(t)\cos(kt) \end{aligned} \tag{5}$$

Performing the trigonometric integrations,

$$\begin{aligned} A &= \frac{2k - \sin(2k)}{4k^2}A + \frac{1-\cos(2k)}{4k^2}B + \alpha \\ B &= \frac{1-\cos(2k)}{4k^2}A + \frac{2k+\sin(2k)}{4k^2}B + \beta \end{aligned} \tag{6}$$

where

$$\alpha = \int_0^1 dt\ c(t)\sin(kt), \qquad \beta = \int_0^1 dt\ c(t)\cos(kt)\ .$$

Since α and β are assumed to be known, we can solve these equations for A and B to get

$$\begin{aligned} A &= \frac{4k^2}{(4k^2-2k)^2 - \sin^2(2k)}\left[\left(4k^2 - 2k - \sin(2k)\right)\alpha + (1-\cos(2k))\beta\right] \\ B &= \frac{4k^2}{(4k^2-2k)^2 - \sin^2(2k)}\left[(1-\cos(2k))\alpha + \left(4k^2 - 2k - \sin(2k)\right)\beta\right] \end{aligned} \tag{7}$$

These set the arbitrary constants in $c(t)$ through (4). Then (3) with A and B as given above gives the solution for this problem with appropriate boundary conditions.

Problem 16

Both inner products are linear in f and g. Also, f and g enter the expression symmetrically. Let's check the last requirement. $\langle f|f\rangle \geq 0$ is obvious for both. If we have $\langle f|f\rangle = 0$ in (b), we have $f = 0$ because f' is continuous:

$$\langle f|f\rangle = \int_0^1 f'(x)f'(x)dx = 0 \Rightarrow f'(x) = 0\ . \tag{8}$$

That means f is constant. If the constant does not happen to be zero, we cannot satisfy the positive definite condition for the inner product.

In the case of (a), $\langle f|f\rangle = 0$ implies both $f' = 0$ and $f(0) = 0$. $f' = 0$ says f is constant but $f(0) = 0$ means f is 0 identically. Therefore, (a) is an inner product.

Problem 17

We could let $|a_i\rangle$ be the column vector with 1 in the i'th position, and zeros in the other spots, and let $\langle b_i|$ be the i'th row of M. Then the sum yields:

$$M = \sum_{i=1}^{n} |a_i\rangle\langle b_i|.$$

Problem 18

(a)

$$|x| = \sum_{n=1}^{\infty} |x_n|^2 = |Cx| \ . \tag{9}$$

Hence $B = 1$ or any number greater than 1 is a good choice.

(b)

If $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are vectors,

$$\begin{aligned} C\,(ax + by) &= C(a(x_1, x_2, \cdots) + b(y_1, y_2, \cdots)) \\ &= C(ax_1 + by_1, ax_2 + by_2, \cdots) \\ &= (0, ax_1 + by_1, ax_2 + by_2, \cdots) \\ &= a(0, x_1, x_2, \cdots) + b(0, y_1, y_2, \cdots) \\ &= aCx + bCy \end{aligned} \tag{10}$$

So, C is a linear operator.

(c)

We have to check if $\langle x|Cy\rangle = \langle Cx|y\rangle$:

$$\begin{aligned} \langle x|Cy\rangle &= \langle (x_1, x_2, \cdots)|(0, y_1, y_2, \cdots)\rangle \\ &= x_2^* y_1 + x_3^* y_2 + \cdots \\ \langle Cx|y\rangle &= \langle (0, x_1, x_2, \cdots)|(y_1, y_2, \cdots)\rangle \\ &= x_1^* y_2 + x_2^* y_3 + \cdots \end{aligned} \tag{11}$$

Therefore, $\langle x|Cy\rangle \neq \langle Cx|y\rangle$. C is not Hermitian.

(d)

$$\begin{aligned} Cx &= 0 \\ (0, x_1, x_2, \cdots) &= (0, 0, 0, \cdots) \end{aligned} \tag{12}$$

The equality holds component-wise. Hence $x_1 = x_2 = \cdots = 0$. There is no non-trivial solution.

(e)

In this case,

$$(0, x_1, x_2, \cdots) = (a_1, a_2, \cdots). \tag{13}$$

If $a_1 \neq 0$, there is no solution. If $a_1 = 0$, the equation is satisfied by setting

$$x_1 = a_2,\ x_2 = a_3, \cdots \tag{14}$$

If a has a finite norm, so does x. Hence there is a solution in this case.

Now, let's consider G. G is a bounded operator since $|Gx| \leq |x|$. Linearity is obvious. Note that

$$\begin{aligned} \langle x|Gy\rangle &= \langle (x_1, x_2, \cdots)|(y_1, y_2/2, \cdots)\rangle \\ &= x_1^* y_1 + x_2^* y_2/2 + \cdots \\ \langle Gx|y\rangle &= \langle (x_1, x_2/2, \cdots)|(y_1, y_2, \cdots)\rangle \\ &= x_1^* y_1 + x_2^* y_2/2 + \cdots \end{aligned} \tag{15}$$

$\langle x|Gy\rangle = \langle Cx|y\rangle$. G is Hermitian.
To have $Gx = 0$, the only possibility is $x_1 = 0, x_2 = 0, \cdots$ by matching the components. So there is only a trivial solution for this equation.
Let's check if $Gx = a$ has a solution.

$$\begin{aligned} Gx &= a \\ (x_1, x_2/2, \cdots) &= (a_1, a_2, \cdots) \\ x_1 &= a_1,\ x_2 = 2a_2, \cdots . \end{aligned} \tag{16}$$

If $a = (a_1, a_2, \cdots) = (1, 1/2, 1/3, \cdots)$, a is normalizable. But $x = (1, 1, 1, \cdots)$, which is not normalizable.

Problem 19

Let's differentiate τ once:

$$\begin{aligned}\frac{d}{dx}\tau &= e^{-\frac{a^2}{(a^2-x^2)}} \\ &= -\frac{2xa^2}{(a^2-x^2)^2}e^{-\frac{a^2}{(a^2-x^2)}}\end{aligned} \tag{17}$$

This tends to 0 as x approaches a due to the exponential factor from the left. By differentiating further, we get a more singular fraction, but still there is an exponential factor $\exp(-\frac{a^2}{(a^2-x^2)})$. Hence any higher order derivative becomes 0 near $x = a$. This means the Taylor series is just 0 around $x = a$. This can be understood by considering τ as a function of a complex variable x. Rename x by z and consider

$$\tau = e^{-\frac{a^2}{a^2-z^2}} \tag{18}$$

on the whole complex plane. When z approaches to a from the right on the real axis, τ diverges. This means that τ has a singularity at $z = a$. Since τ is defined everywhere except at $z = a$, we can make a Laurent expansion

$$\tau = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \ . \tag{19}$$

This can be done simply by expanding the exponential in τ. We get from (18)

$$\tau = 1 - \frac{a^2}{a^2-z^2} + \frac{1}{2}\left(\frac{a^2}{a^2-z^2}\right)^2 - \cdots \tag{20}$$

From this, we can expect that there will be terms with arbitrary negative powers of $z-a$ when we expand a Laurent series. Therefore, the singularity is an essential singularity. It is an isolated singularity because only one point in a complex plane is singular.