Ph129b PS 9 solutions

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37: Parity of a k-cycle

We want to find out the parity of a k-cycle $(n_1 n_2 \cdots n_k)$. We know that any conjugate element of this k-cycle is just a replacement of the original $n_1, n_2, \cdots$ to some other set of numbers, and any other set of numbers can arise by conjugation. In particular, $(n_1 n_2 \cdots n_k)$ and $(12 \cdots k)$ are in the same class. That is

$$(n_1 n_2 \cdots n_k) = S(12 \cdots k)S^{-1} \tag{1}$$

for some permutation S. Note that S and S^{-1} consist of the same number of transpositions. Hence the parity of $(n_1 n_2 \cdots n_k)$ is the same as the parity of $(12 \cdots k)$. But

$$(12 \cdots k) = (12)(23)(34) \cdots (k, k-1) \ . \tag{2}$$

Therefore the parity of a k-cycle is $(-1)^{k-1}$.

You may wonder if this parity is well-defined. That is, what if there are two different sets of decomposition of a k-cycle into compositions of 2-cycles? Of course, there are. But, all other decompositions differ from each other by even number of 2-cycles. To see this, consider a representation of the permutation group S_n where the vector space is $\mathbb{R}^n$ and $(12 \cdots k)$ is represented by

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \ . \tag{3}$$

Other elements are represented in the obvious way. It is easy to check that this indeed is a representation. Any matrix expression of a 2-cycle (k_1, k_2) has its determinant -1 in this representation: $\det(k_1, k_2) = -1$. det is obviously a homomorphism from our representation of S_n to $\mathbb{R}$, and the parity is nothing but this $\mathbb{R}$ representation. Hence the parity of $(12 \cdots k)$ is $\det(12 \cdots k) = (-1)^{k-1}$ and it is unambiguously defined.

38: Properties of S and A

$P_a P$ is just another permutation. $P_a P = P_a P'$ implies $P = P'$. Therefore

$$\sum_P P = P_a \sum_P P \ . \tag{4}$$

That is, $P_a S = S$. From this, it follows that

$$S^2 = \frac{1}{n!}\sum_P PS = \frac{1}{n!}\sum_P S = S \ . \tag{5}$$

Note that $\delta_{P_a}\delta_P = \delta_{P_a P} = \delta_{P P_a}$. Hence

$$P_a \sum_P \delta_P P = \delta_{P_a}\delta_{P_a} P_a \sum_P \delta_P P = \delta_{P_a} \sum_P \delta_{P_a P} P_a P = \delta_{P_a} \sum_P \delta_P P \ . \tag{6}$$

That is, $P_a A = \delta_{P_a} A$. In the same way, we can obtain $A P_a = \delta_{P_a} A$. Finally,

$$A^2 = \frac{1}{n!}\sum_P \delta_P P A = \frac{1}{n!}\sum_P \delta_P \delta_P A = \frac{1}{n!}\sum_P A = A \ . \tag{7}$$

39: A representation of S_3

Under (13),

$$\begin{aligned} \psi_1' &\to -\frac{1}{\sqrt{12}}(2sdu + 2sud - uds - dus - dsu - usd) = -\frac{1}{2}\psi_1' + \frac{\sqrt{3}}{2}\psi_2' \\ \psi_2' &\to -\frac{1}{2}(dsu + usd - uds - dus) = -\frac{\sqrt{3}}{2}\psi_1' + \frac{1}{2}\psi_2' \ . \end{aligned} \tag{8}$$

Therefore

$$D(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ . \tag{9}$$

Other matrices can be found by homomorphism:

$$\begin{aligned} D(123) &= D(13)D(12) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ D(132) &= D(12)D(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ D(23) &= D(132)D(12) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ . \end{aligned} \tag{10}$$

40 Permutation note Ex. 4: Wave function of a proton

It is stated in the problem that the spin-flavor part of the wave function of any $L = 0$ baryon must be symmetric under the exchange of any two of its constituent quarks. This can be achieved by either multiplying two symmetric pieces or two antisymmetric pieces. However, in this particular mixed symmetry representation ($\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$) only two particles can be symmetrized at a time. Picking the first two particles yields the wave functions given in the notes.

$$|p_\uparrow >= \frac{1}{\sqrt{2}}\left(\begin{array}{|c|c|}\hline u & u \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \uparrow & \uparrow \\ \hline \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline d & u \\ \hline u & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \downarrow & \uparrow \\ \hline \uparrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$$

$$|p_\uparrow >= \tfrac{1}{\sqrt{2}}(\chi_+^\lambda \phi_{uud}^\lambda + \chi_+^\rho \phi_{uud}^\rho)$$

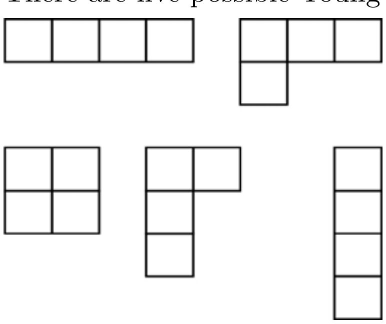
$$\begin{aligned} |p_\uparrow > \quad = \quad & \tfrac{1}{3\sqrt{2}}\,(2|u_\uparrow u_\uparrow d_\downarrow > - |u_\uparrow d_\uparrow u_\downarrow > - |d_\uparrow u_\uparrow u_\downarrow > \\ & -|u_\uparrow u_\downarrow d_\uparrow > + 2|u_\uparrow d_\downarrow u_\uparrow > - |d_\uparrow u_\downarrow u_\uparrow > \\ & -|u_\downarrow u_\uparrow d_\uparrow > - |u_\downarrow d_\uparrow u_\uparrow > + 2|d_\downarrow u_\uparrow u_\uparrow >) \end{aligned}$$

Picking any other pair of particles to symmetrize gives the same final wave function in this particular case. To be safe, you can symmetrize the pairs (12), (23), and (13) separately and then add the answers with equal weights and renormalize. This yields the same answer as above.

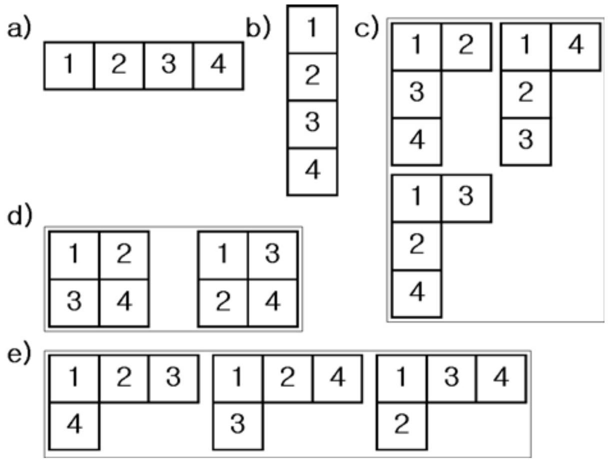
Another way to work this problem is to use the group $SU(4)$ with basis states $u_\uparrow$, $u_\downarrow$, $d_\uparrow$ and $d_\downarrow$ and look at the product rep $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4}$. No one attempted this, although see the Review of Particle Physics summary of the quark model for a brief description using $SU(6)$ (adds the strange quark to the states above): `http://pdg.lbl.gov/2005/reviews/quarkmodrpp.pdf`

41: Young tableaux for S_4

There are five possible Young diagrams:



Each diagram corresponds to an irreducible representation. The number of possible standard Young tableaux for each Young diagram gives the dimension of the representation. The possible standard Young tableaux are



a) and b) correspond to two one dimensional irreducible representations. d) is two dimensional. c) and e) are three dimensional.