


# ph129 PS 1 solutions

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## Problem 1

### part A

Let's prove the convolution theorem for the Fourier transform. Some notation:

$$\mathcal{F}[g(x); y] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ixy} dx \quad (1)$$

Our function  $g(x)$  is given by

$$g(x) = \int_{-\infty}^{\infty} dy f_1(y) f_2(x - y) \quad (2)$$

So we have

$$\mathcal{F}[g(x); z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-izx} \int_{-\infty}^{\infty} dy f_1(y) f_2(x - y) \quad (3)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy f_1(y) e^{-izy} \int_{-\infty}^{\infty} dx f_2(x - y) e^{-iz(x-y)} \quad (4)$$

where the second line follows by regrouping the terms and multiplication by  $1 = e^{izy} e^{-izy}$ . Next, we want to make a change of variables  $u = x - y$  in the second integral, leaving

$$\mathcal{F}[g(x); z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy f_1(y) e^{-izy} \int_{-\infty}^{\infty} du f_2(u) e^{-iuz} \quad (5)$$

$$= \sqrt{2\pi} \mathcal{F}[f_1(x); z] \mathcal{F}[f_2(x); z] \quad (6)$$

## B

We can find  $\hat{N}(y, \mu, \sigma)$  explicitly by doing the following integral:

$$\hat{N}(y, \mu, \sigma) = \int_{-\infty}^{\infty} dx e^{-ixy} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (7)$$

which is not so hard using this general formula for Gaussian integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{c+b^2/4a}. \quad (8)$$

You can prove this formula by completing the square in the exponent, doing a change of variables  $z = x - b/2a$ , and then using the "polar coordinates trick." If you don't know this trick, it goes as follows. Let

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx. \quad (9)$$

Then we can write

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-ax^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-ay^2} dy \right) \quad (10)$$

$$= \int_{-\infty}^{\infty} dx dy e^{-a(x^2+y^2)} \quad (11)$$

$$= \int_0^{\infty} r dr \int_0^{2\pi} d\theta e^{-ar^2} = \frac{\pi}{a} \quad (12)$$

where the last line follows by changing to polar coordinates. Thus we see that we must have  $I = \sqrt{\pi/a}$ .

Another way to evaluate  $\hat{N}$  is to use the results from class on translations. The general result is

$$\mathcal{F}[f(x+a); y] = e^{iay} \mathcal{F}[f(x); y]. \quad (13)$$

Using this we have

$$\hat{N}(y, \mu, \sigma) = \mathcal{F}[N(x, \mu, \sigma); y] = \mathcal{F}[N(x - \mu, 0, \sigma); y] \quad (14)$$

$$= e^{-i\mu y} \hat{N}(y, 0, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-i\mu y} e^{-y^2\sigma^2/2} \quad (15)$$

## C

Let's take the Fourier transform of  $P$  and use our results from parts A and B.

$$\mathcal{F}[P(x); k] = \sqrt{2\pi} \mathcal{F}[N(x, \mu_1, \sigma_1); k] \mathcal{F}[N(y, \mu_2, \sigma_2); k] \quad (16)$$

$$= \sqrt{2\pi} e^{-ik\mu_1} \mathcal{F}[N(x, 0, \sigma_1); k] e^{-ik\mu_2} \mathcal{F}[N(y, 0, \sigma_2); k] \quad (17)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik(\mu_1 + \mu_2)} e^{-k^2(\sigma_1^2 + \sigma_2^2)/2} \quad (18)$$

$$= e^{-ik(\mu_1 + \mu_2)} \mathcal{F}[N(x, 0, \sqrt{\sigma_1^2 + \sigma_2^2}); k] \quad (19)$$

$$= \mathcal{F}[N(x, \mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}); k] \quad (20)$$

This means that

$$P(x) = N(x, \mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left(-\frac{(x - \mu_1 - \mu_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right) \quad (21)$$

## Problem 2

We want to find the momentum-space wave-function  $\Psi(\mathbf{p})$ . In Dirac notation, we have

$$\Psi(\mathbf{p}) = \langle \mathbf{p} | \psi \rangle = \int d^3x \langle \mathbf{p} | \mathbf{x} \rangle \langle \mathbf{x} | \psi \rangle . \quad (22)$$

We know from our quantum mechanics classes that

$$\langle \mathbf{p} | \mathbf{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p} \cdot \mathbf{x}} \quad (23)$$

and position space wave-function is

$$\langle \mathbf{x} | \psi \rangle = \psi(\mathbf{x}) = \frac{1}{\sqrt{32\pi a^5}} r \cos \theta e^{-r/2a} . \quad (24)$$

~~Note that it's slightly different than in Frank's notes, with an additional factor of  $1/a^{3/2}$ . Also, I've set  $\hbar = 1$ , which just means we're measuring momentum in units of (distance)<sup>-1</sup>. So, we have~~

$$\Psi(\mathbf{p}) = \int d^3x \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p} \cdot \mathbf{x}} \frac{1}{\sqrt{32\pi a^5}} r \cos \theta e^{-r/2a} \quad (25)$$

and so we see that Fourier transformation switches back and forth between position and momentum space. Before we dive into the integral, let's choose our coordinates to make the integration as simple as possible. The wave-function is axially symmetric

about  $\hat{z}$ , so let's take  $\mathbf{p}$  to be in the  $x-z$  plane, i.e.  $\mathbf{p} = (p \sin \Omega, 0, p \cos \Omega)$ . Now, let's rotate to a new coordinate system  $(x', y', z')$  so that  $p$  points along  $\hat{z}'$ . The rotation matrix  $\mathbf{R}$  that accomplishes this is

$$\mathbf{R} = \begin{pmatrix} \cos \Omega & 0 & -\sin \Omega \\ 0 & 1 & 0 \\ \sin \Omega & 0 & \cos \Omega \end{pmatrix} \quad (26)$$

and you can verify that  $\mathbf{p}' = \mathbf{R}\mathbf{p} = (0, 0, p)$ . The coordinates in this frame are simply  $\mathbf{x}' = \mathbf{R}\mathbf{x}$ . Now we need to rewrite our spacial wave-function in terms of the new coordinates. First, note that  $r$  is invariant under rotations. Next, we know that  $r \cos \theta = z$ , and so from the inverse rotation  $\mathbf{x} = \mathbf{R}^{-1}\mathbf{x}'$ , we have

$$z = x' \sin \Omega + z' \cos \Omega = r(\sin \alpha \cos \beta \sin \Omega + \cos \alpha \cos \Omega) \quad (27)$$

where  $\alpha$  and  $\beta$  are the polar and azimuthal angles in the primed coordinate frame. Lastly, since scalar products are invariant under rotations, we see that

$$\mathbf{p} \cdot \mathbf{x} = \mathbf{p}' \cdot \mathbf{x}' = pz' = p \cos \alpha \quad (28)$$

Now finally we can write down the full integral in the primed frame:

$$\Psi(\mathbf{p}) = \int_0^\infty r^2 dr \int_0^\pi \sin \alpha d\alpha \int_0^{2\pi} d\beta \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{32\pi a^5}} r(\sin \alpha \cos \beta \sin \Omega + \cos \alpha \cos \Omega) \quad (29)$$

$$\times \exp((-ip \cos \alpha - 1/2a)r) \quad (30)$$

Now, this is an integral we can do pretty easily. First, note that  $\int d\beta \cos \beta = 0$  in the first term, while  $\int d\beta = 2\pi$  in the second term. Next, we'll do the  $\int d\alpha$  integral by making a substitution  $\mu = \cos \alpha$ . Using integration by parts, we can show that

$$\int_{-1}^1 d\mu \mu e^{A\mu} = \frac{2A \cosh A - 2 \sinh A}{A^2}. \quad (31)$$

Then, after some simplification, we get

$$\Psi(\mathbf{p}) = \int_0^\infty r dr \frac{i \cos \Omega}{2\pi a^{5/2} p^2} e^{-r/2a} (pr \cos pr - \sin pr). \quad (32)$$

These integrals can be evaluated by writing sine and cosine as complex exponentials and then using integration by parts. When the dust settles, we have

$$\Psi(\mathbf{p}) = \frac{8i \cos \Omega p a^{5/2}}{\pi(4a^2 p^2 + 1)^3}. \quad (33)$$

[Note: the 128 factor should be 64]

Our final result is still axially symmetric about  $\hat{z}$ , just as our original spacial wave-function. Finally note what happens to both wave-functions when you increase or decrease  $a$ . For example, increasing  $a$  causes the spacial wave-function to spread out in space. The electron is less localized around its nucleus and the uncertainty in its position increases. However, the momentum-space wave-function becomes more sharply peaked and more localized. The uncertainty in the electron's momentum decreases. You can make a few plots in Mathematica for various values of  $a$  to convince yourself. This is the uncertainty principle in action!

### Problem 3

Our voltage pulse  $V(t)$  is the sum of two pulses  $V_1(t) = Ae^{-t/\tau_1}$  and  $V_2(t) = -Ae^{-t/\tau_2}$ . Because the rules that govern circuits are linear, we know that the superposition principle applies and so we can solve for each  $V_C$  from each pulse individually, and then add the results together. So let's just solve the slightly simpler case  $V(t) = Ae^{-t/\tau}$ . First let's take the Laplace transform:

$$\hat{V}(s) = \int_0^\infty dt e^{-st} V(t) = \frac{A\tau}{s\tau + 1} \quad (34)$$

Next, we know from class that

$$V_C(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds e^{st} \frac{R_2}{R_1 + R_2 + sR_1R_2C} \hat{V}(s) \quad (35)$$

$$= \frac{A}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds e^{st} \frac{R_2}{R_1 + R_2 + sR_1R_2C} \frac{\tau}{s\tau + 1}. \quad (36)$$

We can do this integral by contour integration, as done in class. There are two poles at  $s = -1/\tau$  and  $s = -1/B = -(R_1 + R_2)/(R_1R_2C)$ . The residues of the integrand are given by

$$\text{Res}[s = -1/\tau] = e^{-t/\tau} \frac{B\tau}{R_1C(\tau - B)} \quad (37)$$

$$\text{Res}[s = -1/B] = e^{-t/B} \frac{B\tau}{R_1C(B - \tau)} \quad (38)$$

And so we get

$$V_C(t) = \frac{AB\tau}{R_1C(B - \tau)} \left( e^{-t/B} - e^{-t/\tau} \right). \quad (39)$$

We can compare this with the result from class by taking the limit in which our potential  $V(t) \rightarrow \delta(t)$ , i.e.  $A \rightarrow \infty$ ,  $\tau \rightarrow 0$ , and  $A\tau \rightarrow 1$ . (Remember that  $V(t)$  vanishes for  $t < 0$ .) In this limit,  $e^{-t/\tau} \rightarrow 0$ , and we get

$$V_C(t) = \frac{1}{R_1 C} e^{-t/B} \quad (40)$$

which agrees with the class notes. We can also take the limit that  $\tau \rightarrow \infty$ , which corresponds to applying a constant voltage. In this case, after the transient  $e^{-t/B}$  term decays away, we are left with  $V_C = AR_2/(R_1 + R_2)$ , which is exactly what you would find from an intro-EM analysis of this circuit with constant voltages. Now we can easily form the linear combination of pulses  $V_1$  and  $V_2$ :

$$V_C(t) = \frac{A B \tau_1}{R_1 C (B - \tau_1)} \left( e^{-t/B} - e^{-t/\tau_1} \right) - \frac{A B \tau_2}{R_1 C (B - \tau_2)} \left( e^{-t/B} - e^{-t/\tau_2} \right). \quad (41)$$

**Problem 4** [Note that the subscripts are re-labeled from the problem statement, A  $\rightarrow$  1, etc]

Recall that the rule for taking the Laplace transform of a derivative is

$$\mathcal{L}[f'(x)] = s \mathcal{L}[f(x)] - f(0). \quad (42)$$

Then if we take the Laplace transform of our system of equations, we find

$$s \hat{N}_1 - N = -\lambda_1 \hat{N}_1 \quad (43)$$

$$s \hat{N}_2 = \lambda_1 \hat{N}_1 - \lambda_2 \hat{N}_2 \quad (44)$$

$$s \hat{N}_3 - n = \lambda_2 \hat{N}_2 - \lambda_3 \hat{N}_3. \quad (45)$$

We can solve the first equation right away. It yields

$$\hat{N}_1(s) = \frac{N}{\lambda_1 + s}. \quad (46)$$

We can invert this to find  $N_1$  as follows

$$N_1(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{N}_1(s) e^{st} ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{N}{\lambda_1 + s} e^{st} ds. \quad (47)$$

We can evaluate this by contour integration as we did for problem 3, i.e. closing the contour at  $-\infty$  and picking up the pole at  $s = -1/\lambda_1$ . The result is

$$N_1(t) = N e^{-\lambda_1 t} \quad (48)$$

just as we would expect from a simple nuclear decay problem. Next, we can solve for  $N_2$ . Plugging in for  $\hat{N}_1$ , we see that

$$\hat{N}_2(s) = \frac{N \lambda_1}{(s + \lambda_1)(s + \lambda_2)} \quad (49)$$

and thus we have

$$N_2(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{N\lambda_1}{(s+\lambda_1)(s+\lambda_2)} e^{st} ds. \quad (50)$$

We pick up two poles, at  $s = -1/\lambda_{1,2}$ , and get

$$N_2(t) = e^{-\lambda_1 t} \frac{N\lambda_1}{\lambda_2 - \lambda_1} - e^{-\lambda_2 t} \frac{N\lambda_1}{\lambda_1 - \lambda_2}. \quad (51)$$

Note that if  $\lambda_2 = 0$ , then  $N_2(t) = N(1 - e^{-\lambda_1 t})$ , which makes sense. Finally, we plug  $\hat{N}_2$  into the equation for  $\hat{N}_3$  and find

$$\hat{N}_3(s) = \frac{n}{s + \lambda_3} + \frac{N\lambda_1\lambda_2}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)} \quad (52)$$

and so we get

$$N_3(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{n}{s + \lambda_3} + \frac{N\lambda_1\lambda_2}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)} \right) e^{st} ds \quad (53)$$

$$= e^{-\lambda_1 t} \left( \frac{N\lambda_1\lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \right) + e^{-\lambda_2 t} \left( \frac{N\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \right) \quad (54)$$

$$+ e^{-\lambda_3 t} \left( n + \frac{N\lambda_1\lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right). \quad (55)$$