Ph129 PS 4 solutions

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Problem 14 (Integral Equations note, Exercise 19)

We can write the solution to an integral equation with a symmetric kernel as follows:

$$f(x) = g(x) + \lambda \sum_n \frac{< \beta_n | g >}{\lambda_n - \lambda} \beta_n(x) \tag{1}$$

where

$$< \beta_n | g >= \int_0^{2\pi} dx\, \beta_n(x) g(x). \tag{2}$$

The functions $\beta_n(x)$ are the normalized eigenfunctions of the kernel, with eigenvalue λ_n, and the sum runs over every eigenvalue and eigenfunction. In discussed in class, the eigenfunctions are

$$\beta_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \tag{3}$$

with eigenvalues $\lambda_n = \alpha^{-|n|}$. Note that n can be any integer, so the sum runs over all integers, from $-\infty$ to ∞. Now we have $g(x) = e^x$, so

$$< \beta_n | g >= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} dx e^{(1-in)x} = \frac{e^{2\pi} - 1}{1 - in} \,. \tag{4}$$

Note that here $< \beta_n | g > \neq 0$ for any value of n, and so there is no solution for $\lambda = \lambda_n$ for any n. This is different than in the example in class. In class, we had $< \beta_n | g >= 0$ unless $n = -2, 0, 2$, and there existed a solution for $\lambda = \lambda_n$ except for those particular values of n. And so we have

$$f(x) = e^x + \frac{\lambda}{2\pi} \sum_n \frac{e^{2\pi} - 1}{1 - in} \frac{e^{inx}}{\lambda_n - \lambda} \ . \tag{5}$$

This series converges. This can be shown, for example, by using the ratio test:

$$\left| \frac{e^{2\pi} - 1}{1 - i(n+1)} \frac{e^{i(n+1)x}}{\lambda_{n+1} - \lambda} \right| / \left| \frac{e^{2\pi} - 1}{1 - in} \frac{e^{inx}}{\lambda_n - \lambda} \right| \to \left| \frac{\lambda_n}{\lambda_{n+1}} \right| = \alpha < 1 \qquad as \qquad n \to \infty \tag{6}$$

Problem 15 (Hilbert Spaces Exercise 1)

First, some remarks about the meaning "neighborhood". Let t be a point in $\mathcal{T}$, let O_t be an open set that contains t, and let N_t be any kind of set that contains O_t. So, the hierarchy goes $t \in O_t \subseteq N_t \subseteq \mathcal{T}$. Then, according to the definition in class, N_t is a neighborhood of t, since it contains an open set O_t that contains t. Of course, O_t is also a neighborhood of t, since O_t contains itself, and so we can just consider our neighborhoods on $\mathcal{T}$ to be open neighborhoods.

Now on the with the proof. We have two statements:

(1) A is an open set.

(2) A contains a neighborhood of each of its points.

We want to prove (1) $\Leftrightarrow$ (2). The proof has two parts: first show that (1) $\Rightarrow$ (2), and then show that (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let A be an open set. Then, clearly, A is a neighborhood of each point $a \in A$. Since A contains itself, A contains a neighborhood of each of its points.

(2) $\Rightarrow$ (1): We want to suppose that, for any point $a \in A$, there exists $N_a \subseteq A$, an open neighborhood of a. Let's define a new set $B = \bigcup_a N_a$, that is, B is the union of the neighborhoods N_a of all points $a \in A$. We can see that we must have $B = A$. First, B cannot contain any points outside of A, since all the neighborhoods N_a which compose B are each contained in A. Second, A cannot contain any points outside of B, since the union sums over all points in A. Since $A = B$, A is a union of open sets and therefore must be open itself.

Problem 16 (Exercise 2 in Hilbert Spaces note)

(a) It satisfies all the definition of topological space.

(b) It is not topological space, because, for example, intersection of $\{1,2,3,4,5\}$ and $\{5,6\}$ is $\{5\}$ but it is not in τ.

(c) It is not topological space, because $\mathcal{T}$ is not in τ.

(d) It is topological space. Intersection between non-intersecting closed intervals give empty set, and union of infinitely many closed intervals, for example, $\bigcup_{n=1}^{\infty}[-n,n]$ give $\mathcal{R}$. It also satisfies other requirements for being topological space.

(e) It is topological space. We have n^2 number of independent real numbers which we get from $n+2\frac{n(n-1)}{2}$, where n comes from diagonal part and $\frac{n(n-1)}{2}$ from upper triangle part without diagonals and 2 from real and imaginary part of the element. We can think it as product topology on $\mathcal{R}^{n^2}$ with the usual topology on each $\mathcal{R}$.

(f) It is topological space. It is similar to the case of (e). We have three independent real numbers, so we can think it as the product topology on $\mathcal{R}^3$ with the usual topology on $\mathcal{R}$. By the way, considering the periodicity of rotations, we can restrict the Euler angles to $\phi \in [0,2\pi), \theta \in [0,\pi], \psi \in [0,2\pi)$ of $\mathcal{R}^3$. Then we can think it as product and subspace topology. Given the usual topology on $\mathcal{R}$, we make subspace topology with basis which is intersections of, for example, $[0,2\pi)$ and open intervals in $\mathcal{R}$. Then we take cartesian product of them.

Problem 17 (Exercise 11 in Integral Equations note)

We will use a method similar to what we used for degenerate kernels. Expanding $\sin k(s-t)$ in the equation,

$$\frac{d^2u}{dt^2}(t) + A\cos(kt) - B\sin(kt) = a(t) \tag{7}$$

where

$$A = \int_0^1 u(s)\sin(ks)ds, \qquad B = \int_0^1 u(s)\cos(ks)ds\ . \tag{8}$$

Integrating (7) twice,

$$u(t) = \frac{A}{k^2}\cos(kt) - \frac{B}{k^2}\sin(kt) + c(t) \tag{9}$$

where

$$c(t) = \iint a(s) ds \ .$$

For an indefinite integration, we need to specify one condition to fix an arbitrary constant. Since $c(t)$ is given by double integration, two conditions should be imposed on $c(t)$. Using $u(0) = 0$ and $u'(0) = 0$,

$$\begin{aligned} c(0) &= -\frac{A}{k^2} \\ c'(0) &= \frac{B}{k} \end{aligned} \tag{10}$$

These two conditions fix $c(t)$ completely once we have the values of A and B. Using the expression for $u(t)$ in (9), (8) says

$$\begin{aligned} A &= \frac{A}{k}\int_0^1 dt\ \sin^2(kt) + \frac{B}{k}\int_0^1 dt\ \sin(kt)\cos(kt) + \int_0^1 dt\ c(t)\sin(kt) \\ B &= \frac{A}{k}\int_0^1 dt\ \sin(kt)\cos(kt) + \frac{B}{k}\int_0^1 dt\ \cos^2(kt) + \int_0^1 dt\ c(t)\cos(kt) \end{aligned} \tag{11}$$

Performing the trigonometric integrations,

$$\begin{aligned} A &= \frac{2k - \sin(2k)}{4k^2} A + \frac{1 - \cos(2k)}{4k^2} B + \alpha \\ B &= \frac{1 - \cos(2k)}{4k^2} A + \frac{2k + \sin(2k)}{4k^2} B + \beta \end{aligned} \tag{12}$$

where

$$\alpha = \int_0^1 dt\ c(t)\sin(kt), \qquad \beta = \int_0^1 dt\ c(t)\cos(kt) \ .$$

Since α and β are assumed to be known, we can solve these equations for A and B to get

$$\begin{aligned} A &= \frac{4k^2}{(4k^2 - 2k)^2 - \sin^2(2k)} \left[\left(4k^2 - 2k - \sin(2k)\right)\alpha + (1 - \cos(2k))\beta\right] \\ B &= \frac{4k^2}{(4k^2 - 2k)^2 - \sin^2(2k)} \left[(1 - \cos(2k))\alpha + \left(4k^2 - 2k - \sin(2k)\right)\beta\right] \end{aligned} \tag{13}$$

These set the arbitrary constants in $c(t)$ through (10). Then (9) with A and B as given above gives the solution for this problem with appropriate boundary conditions.

Problem 18 (Exercise 5 in Hilbert Spaces note)

Both inner products are linear in f and g. Also, f and g enter the expression symmetrically. Let's check the last requirement. $\langle f|f\rangle \geq 0$ is obvious for both. If we have $\langle f|f\rangle = 0$ in (b), we have $f = 0$ because f' is continuous:

$$\langle f|f\rangle = \int_0^1 f'(x)f'(x)dx = 0 \Rightarrow f'(x) = 0\ . \tag{14}$$

That means f is constant. If the constant does not happen to be zero, we cannot satisfy the positive definite condition for the inner product.

In the case of (a), $\langle f|f\rangle = 0$ implies both $f' = 0$ and $f(0) = 0$. $f' = 0$ says f is constant but $f(0) = 0$ means f is 0 identically. Therefore, (a) is an inner product.