

Ph 135b: Solution Set 5

February 19, 2004

24

There are two cases to consider i) $v_p > v_b$ ii) $v_p < v_b$.

Case i) Imagine the particle enters the box from the right. If the ball is already moving to the right they will collide with relative velocity $v_p + v_b$. The ball spends half its time moving in this direction. If the ball is moving to the left but will bounce off the wall and change direction before the particle collides with it the relative velocity will again be $v_p + v_b$. This happens if the ball is within a distance $x \leq L \frac{v_b}{v_p}$, with L being the length of the box. So the collision will occur with velocity $v_p + v_b$ with probability,

$$P(v_p + v_b) = \frac{L + L \frac{v_b}{v_p}}{2L} \quad (1)$$

$$= \frac{1}{2} \left(1 + \frac{v_b}{v_p} \right). \quad (2)$$

Case ii) Here the particle will never catch the ball before it strikes the wall and turns around so the collision always occurs with relative velocity $v_p + v_b$.

We can consider the deuteron as a box with the proton and neutron moving inside. The internal velocities of the nucleons are much greater than that of the deuteron i.e. the lab frame is almost the same as the rest frame of the deuteron. In this frame the proton and neutron are always moving in opposite directions. So when the pion and neutron collide, which occurs most often head-on, the proton is moving along the same direction as the pion and continues in this direction after the collision. So it moves along a direction with $-\pi/2 < \theta < \pi/2$ or $\cos \theta > 0$.

25

a) This part was apparently done in class but it is straight forward to see that given,

$$\gamma^\mu p_\mu - mc = 0, \quad (3)$$

$$\Rightarrow \gamma^0 p_0 = \gamma^i p_i + mc, \quad (4)$$

$$\Rightarrow H = cp_0 = c\gamma^0(\gamma^i p_i + mc). \quad (5)$$

b)

$$[H, L_i] = [H, \epsilon^{ijk} r_j p_k] \quad (6)$$

$$= c\gamma^0 [\gamma^l p_l, \epsilon^{ijk} r_j p_k] \quad (7)$$

$$= c\gamma^0 \gamma^l \epsilon^{ijk} p_k \delta_{l,j} (-i\hbar) \quad (8)$$

$$= -i\hbar c\gamma^0 (\vec{\gamma} \times \vec{p})_i. \quad (9)$$

c)

$$[H, S^i] = [H, \hbar/2\Sigma^i] \quad (10)$$

$$= mc^2\hbar/2[\gamma^0, \Sigma^i] + c\hbar/2[\gamma^0\gamma^j, \Sigma^i]p_j. \quad (11)$$

Now,

$$[\gamma^0, \Sigma^i] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (12)$$

$$= \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad (13)$$

$$= 0. \quad (14)$$

and similarly

$$[\gamma^0\gamma^j, \Sigma^i] = \begin{pmatrix} 0 & 2i\epsilon^{jik}\sigma^k \\ 2i\epsilon^{jik}\sigma^k & 0 \end{pmatrix}. \quad (15)$$

So

$$[H, S^i] = i\hbar \begin{pmatrix} 0 & -\vec{p} \times \vec{\sigma} \\ -\vec{p} \times \vec{\sigma} & 0 \end{pmatrix} \quad (16)$$

$$= i\hbar\gamma^0(\vec{\gamma} \times \vec{p}). \quad (17)$$

Hence we see that $\vec{J} = \vec{L} + \vec{S}$ is conserved.

d)

$$S^2 = \frac{\hbar^2}{4} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \quad (18)$$

but $\sigma^2 = 3I$ so that

$$S^2 = \hbar^2 3/4 \quad (19)$$

or $s = 1/2$.

26

Under the transformation $\Psi \rightarrow \Psi' = S\Psi$ we have

$$\bar{\Psi}\gamma^5\Psi \rightarrow \bar{\Psi}'\gamma^5\Psi' \quad (20)$$

$$= \Psi^\dagger S^\dagger \gamma^0 \gamma^5 S \Psi. \quad (21)$$

Now,

$$S^\dagger \gamma^0 \gamma^5 S = (a_+ + a_-(\gamma^1)^\dagger(\gamma^0)^\dagger)\gamma^0\gamma^5(a_+ + a_-\gamma^1\gamma^0) \quad (22)$$

$$= (a_+ + a_-\gamma^1\gamma^0)\gamma^0\gamma^5(a_+ + a_-\gamma^1\gamma^0) \quad (23)$$

$$= a_+^2(\gamma^0\gamma^5) - a_+a_-(0) - a_-^2(\gamma^0\gamma^5) \quad (24)$$

$$= \gamma^0\gamma^5. \quad (25)$$

which gives us the required result.

27

If we consider p and n as two states of a single fermionic nucleon then wavefunction describing two nucleons must be totally antisymmetric under particle exchange. So

$$I = 0 \Rightarrow \begin{cases} S = 0 \Rightarrow L = \text{odd} \\ S = 1 \Rightarrow L = \text{even} \end{cases} \quad (26)$$

as a spin 0 state is antisymmetric and a spin 1 state symmetric. Similarly

$$I = 1 \Rightarrow \begin{cases} S = 0 \Rightarrow L = \text{even} \\ S = 1 \Rightarrow L = \text{odd} \end{cases} \quad (27)$$

This can be combined into saying that $L + S + I$ must be odd. Now we know that the deuteron is in an $I = 0$ state as we do not find any pp or nn bound states. We also know that the parity is even so that L must be even. Hence S must be 1. Even though orbital angular momentum is not a good quantum number for the interaction we expect that as the deuteron has only one bound state that its wavefunction has a large admixture of $L = 0$ and so the total angular momentum, which is a good quantum number, should be one.

28

We know that as E are real eigenvalues and the energy eigenvectors form a basis H must be Hermitian. Since \vec{p} is Hermitian we have

$$H = H^\dagger \quad (28)$$

$$\Rightarrow \vec{\alpha} \cdot \vec{p} + \alpha_0 m = \vec{\alpha}^\dagger \cdot \vec{p} + \alpha_0^\dagger m \quad (29)$$

$$\Rightarrow \vec{\alpha} \text{ and } \alpha_0 \text{ are Hermitian} \quad (30)$$

Using the fact that $H^2\Psi = E^2\Psi$ we get

$$H^2\Psi = (\alpha^i \alpha^j p_i p_j + \alpha_0^2 m^2 + m\{\alpha^i, \alpha_0\} p_i)\Psi \quad (31)$$

$$= (p_i^2 + m^2)\Psi. \quad (32)$$

which implies that

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \{\alpha_0, \alpha^j\} = 0, \alpha_0^2 = 1. \quad (33)$$

Now

$$tr(\alpha^i) = tr(\alpha_0 \alpha_0 \alpha_i) = -tr(\alpha_0 \alpha_i \alpha_0) \quad (34)$$

$$= -tr(\alpha_0 \alpha_0 \alpha_i) = -tr(\alpha_i) = 0. \quad (35)$$

$$(36)$$

where we used the above anticommutation relation and the fact that $tr(AB) = tr(BA)$. We also have $tr(\alpha_0) = 0$ by a similar calculation.

As all α matrices satisfy $\alpha^2 = 1$ they all must have eigenvalues of ± 1 . Choose any particular matrix and diagonalise it. It now has only ± 1 's on the diagonal. However these sum to zero so there must be the same number of $+1$'s and -1 's i.e. the matrix is even dimensional.

The number of free parameters in an $n \times n$ complex matrix is $2n^2$. If we insist that the matrix is hermitian we get $n(n-1) - n$ constraints. If we insist that the matrix is traceless we get 1 more constraint. We want the number of free parameters to be 4. So

$$2n^2 - n^2 - 1 \geq 4 \quad (37)$$

$$\Rightarrow n \geq 3 \quad (38)$$

but because n must be even $n \geq 4$.

29

a) Using $\gamma^\mu = (\alpha_0, \alpha_0 \vec{\alpha})$ we have that,

$$(\gamma^i)^\dagger = (\alpha_0 \alpha^i)^\dagger \quad (39)$$

$$= (\alpha^i)^\dagger (\alpha_0)^\dagger \quad (40)$$

$$= \alpha^i \alpha_0 \quad (41)$$

$$= \alpha_0 \alpha_0 \alpha^i \alpha_0 \quad (42)$$

$$= \gamma^0 \gamma^i \gamma^0 \quad (43)$$

as required. Also

$$(\gamma^0)^\dagger = (\alpha_0)^\dagger \quad (44)$$

$$= \alpha_0 \quad (45)$$

$$= \alpha_0 \alpha_0 \alpha_0 \quad (46)$$

$$= \gamma^0 \gamma^0 \gamma^0. \quad (47)$$

b) From the dirac equation we have

$$(\gamma^\mu p_\mu - m)\Psi = 0 \quad (48)$$

now we act with $(\gamma^\nu p_\nu - m)$ on both sides and get,

$$\Rightarrow (\gamma^\nu \gamma^\mu p_\nu p_\mu - m^2)\Psi = 0 \quad (49)$$

We require that $(g^{\nu\mu} p_\nu p_\mu - m^2)\Psi = 0$. By comparing the two equations we get that $\{\gamma^\nu, \gamma^\mu\} = 2g^{\nu\mu}$ as required. We can get the same result by using the definition of the γ 's in terms of the α 's and the anticommutation relations from the last question.