Ph 135b: Solution Set 5

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a)

$$\not{a}\not{b} + \not{b}\not{a} = a^\mu b^\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \quad (1)$$
$$= a^\mu b^\nu(2g^{\mu\nu}) \quad (2)$$
$$= 2a \cdot b \quad (3)$$

b)

$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = \gamma_\mu\gamma^\alpha(2g^{\beta\mu} - \gamma^\mu\gamma^\beta) \quad (4)$$
$$= 2\gamma^\beta\gamma^\alpha - \gamma_\mu(2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)\gamma^\beta \quad (5)$$
$$= 2\gamma^\beta\gamma^\alpha + 4\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta \quad (6)$$
$$= 4g^{\alpha\beta} \quad (7)$$

c)

$$Tr(\gamma^\mu) = 0 \quad (8)$$

by definition. Consider 3 gamma matrices with different indicies (if two are the same we just get the case above),

$$Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) = Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}(\gamma^5)^2) \quad (9)$$
$$= Tr(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^5), \text{ by cyclic permutation} \quad (10)$$
$$= -Tr((\gamma^5)^2\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}), \text{by commuting} \quad (11)$$
$$= -Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}) \quad (12)$$
$$= 0. \quad (13)$$

For higher numbers of indicies two or more of these must be the same and so the trace just reduces to one of the above. d)

$$Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = Tr(\gamma^{\mu_1}\gamma^{\mu_2}(2g^{\mu_3\mu_4} - \gamma^{\mu_4}\gamma^{\mu_3})) \quad (14)$$
$$= 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} - Tr(\gamma^{\mu_1}(2g^{\mu_2\mu_4} - \gamma^{\mu_4}\gamma^{\mu_2})\gamma^{\mu_3}) \quad (15)$$
$$= 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_4} + Tr((2g^{\mu_1\mu_4} - \gamma^{\mu_4}\gamma^{\mu_1})\gamma^{\mu_1}\gamma^{\mu_3}) \quad (16)$$
$$= 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_4} + 8g^{\mu_1\mu_4}g^{\mu_2\mu_3} - Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) \quad (17)$$
$$\Rightarrow Tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_4} + 8g^{\mu_1\mu_4}g^{\mu_2\mu_3} \quad (18)$$

e) Using the above result and the fact that $\gamma^5 \propto \gamma^0\gamma^1\gamma^2\gamma^3$ we see that $Tr(\gamma^5) = 0$.

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a)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\lambda = i\hbar\kappa a(-p_0^2/c^2\hbar + |\vec{p}|^2/\hbar) \quad (19)$$

$$= 0 \quad (20)$$

as the photon is a massless particle.

b)

$$A'_\mu = A_\mu + \partial_\mu \lambda \quad (21)$$

so

$$A'_\mu = ae^{\frac{-ip\cdot x}{\hbar}}\epsilon^\mu(p) + i\hbar\kappa a e^{\frac{-ip\cdot x}{\hbar}}(-ip^\mu/\hbar) \quad (22)$$

$$\Rightarrow A'_\mu = ae^{\frac{-ip\cdot x}{\hbar}}(\epsilon^\mu(p) + p^\mu\kappa) \quad (23)$$

as required.

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a)Using the Feynman rules and going backwards along the fermion lines of the figure on pg 260 we get,

$$-i(2\pi)^4\delta(p_1+p_2-p_3-p_4)\mathcal{M} = \int \frac{d^4q}{(2\pi)^4}[\bar{v}(p_2)ig_e\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)iQg_e\gamma^\nu v(p_4)] \quad (24)$$

$$\times(2\pi)^4\delta^4(p_1+p_2-q)(2\pi)^4\delta^4(p_1+p_2-q) \quad (25)$$

$$\Rightarrow \mathcal{M} = \frac{-Qg_e^2}{(p_1+p_2)^2}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)] \quad (26)$$

where in the second vertex we included a factor of Q to account for the quarks charge and there is an overall irrelevant minus sign.

b)Summing over final states and averaging over initial states we have,

$$<|\mathcal{M}|^2> = \left[\frac{Qg_e^2}{4(p_1+p_2)^2}\right]^2 \sum_{\text{spins}}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)][\bar{v}(p_2)\gamma^\nu u(p_1)]^*[\bar{u}(p_3)\gamma_\nu v(p_4)]^* \quad (27)$$

$$= \left[\frac{-Qg_e^2}{4(p_1+p_2)^2}\right]^2 \sum[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)][\bar{u}(p_1)\gamma^\nu v(p_2)][\bar{v}(p_4)\gamma_\nu u(p_3)] \quad (28)$$

using using the fact that

$$[\bar{v}(a)\gamma^\mu u(b)]^* = \bar{u}(b)\gamma^0(\gamma^\mu)^\dagger\gamma^0 v(a) \quad (29)$$

$$= \bar{u}(b)\gamma^\mu v(a). \quad (30)$$

With $\sum u(p)\bar{u}(p) = \not{p} + mc$ and $\sum v(p)\bar{u}(p) = \not{p} - mc$ we get,

$$<|\mathcal{M}|^2> = \left[\frac{Qg_e^2}{4(p_1+p_2)^2}\right]^2 Tr[\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_2-m)]Tr[\gamma_\mu(\not{p}_4-M)\gamma_\nu(\not{p}_3+m)] \quad (31)$$

c) Now we use the fact that $Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, $Tr(\gamma^\mu \not{p} \gamma^\nu) = 0$ as the trace of an odd number of gamma matrices vanishes and finally,

$$Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) = 4(p_1^\mu p_2^\nu - 2g^{\mu\nu} p_1 \cdot p_2 + p_1^\nu p_2^\mu) \tag{32}$$

so we get

$$\begin{aligned} <|\mathcal{M}|^2> &= \left[\frac{Qg_e^2}{4(p_1+p_2)^2}\right]^2 [-4(mc)^2 g^{\mu\nu} + 4(p_1^\mu p_2^\nu - 2p_1 \cdot p_2 g^{\mu\nu} + p_1^\nu p_2^\mu)] && (33) \\ &\quad \times[-4(Mc)^2 g_{\mu\nu} + 4((p_3)_\mu (p_4)_\nu - 2p_3 \cdot p_4 g_{\mu\nu} + (p_3)_\nu (p_4)_\mu)] && (34) \end{aligned}$$

which when simplified gives the required answer.

d)We go to the c.o.m. frame and choose

$$\begin{aligned} p_1 = (E, 0, 0, p) &\qquad p_2 = (E, 0, 0, -p) && (35) \\ p_3 = (E, k\sin\theta, 0, k\cos\theta) &\qquad p_4 = (E, -k\sin\theta, 0, -k\cos\theta) && (36) \end{aligned}$$

with $p^2c^2 = E^2 - m^2c^4$ and $k^2c^2 = E^2 - M^2c^4$ so,

$$\begin{aligned} <|\mathcal{M}|^2> &= 8\left[\frac{Qg_e^2}{(2E)^2}\right]^2 [(E^2 - c^2pk\cos\theta)^2 + (E^2 + c^2pk\cos\theta)^2 && (37) \\ &\quad +(mc^2)^2(E^2+k^2) + (Mc^2)^2(E^2+p^2) + 2c^2(mc)^2(Mc)^2] && (38) \\ &= [Qg_e^2]^2\left[1 + \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\cos^2\theta + \frac{m^2c^4}{E^2} + \frac{M^2c^4}{E^2}\right] && (39) \end{aligned}$$

which is the required answer.

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We have the formula

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\bar{\mathcal{M}}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|} && (40) \\ &= \left(\frac{\hbar c}{8\pi}\right)^2 [Qg_e^2]^2\left[1 + \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\cos^2\theta + \frac{m^2c^4}{E^2} + \frac{M^2c^4}{E^2}\right] && (41) \end{aligned}$$

now we want to integrate over all angles. The part of the expression indepentdent of $\cos\theta$ just gets an overall 4π and we have that $\int_0^\pi \cos^2\theta d(\cos\theta) = 2/3$ giving

$$\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 [Qg_e^2]^2\left[4\pi(1 + \frac{m^2c^4}{E^2} + \frac{M^2c^4}{E^2}) + 4\pi/3\left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\right] \tag{42}$$

which now simplifies to the correct answer.

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$$\begin{aligned} u(x) &= u_v(x) + s(x) && (43) \\ \bar{u}(x) &= s(x) && (44) \end{aligned}$$

so $\int_0^1 dx\ u(x) - \bar{u}(x) = \int_0^1 dx\ u_v(x) = 2$.Similarly $\int_0^1 dx\ d(x) - \bar{d}(x) = 1$ and $\int_0^1 dx\ s(x) - \bar{s}(x) = 0$.