Ph 135b: Solution Set 7

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Starting from the two diagrams below we get for the amplitude,

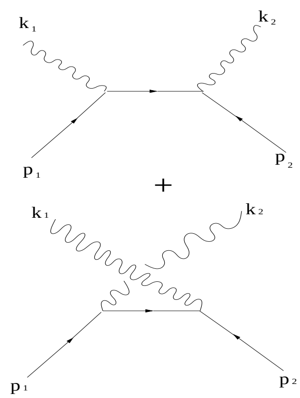

Figure 1: Feyman Diagrams

$$i\mathcal{M} = \bar{v}(p_2)(-ig\gamma^\mu)\epsilon^*_\mu(k_2)\frac{(i(\not{p}_1 - \not{k}_1)}{(p_1-k_1)^2}(-ig\gamma^\nu)\epsilon^*_\nu(k_1)u(p_2) \quad (1)$$

$$+\bar{v}(p_2)(-ig\gamma^\mu)\epsilon^*_\mu(k_1)\frac{(i(\not{p}_1 - \not{k}_2)}{(p_1-k_2)^2}(-ig\gamma^\nu)\epsilon^*_\nu(k_2)u(p_1). \quad (2)$$

where we have set $m = 0$ which implies for example that $(p_1 - k_1)^2 = -2p_1 \cdot k_1$. Now we multiply by the complex conjugate and sum over spins and polarizations. We use the replacement $\sum \epsilon^*_\mu(k_1)\epsilon_\nu(k_1) \to -g_{\mu\nu}$. We also need the fact that

$$\bar{\Gamma} = \gamma^0(\gamma^\mu\gamma^\nu\gamma^\rho)^\dagger\gamma^0 \quad (3)$$

$$= \gamma^\rho\gamma^\nu\gamma^\mu \quad (4)$$

so that after summing over spins and averaging we get

$$< |\mathcal{M}|^2 > = \frac{g^4}{4} tr\left(\not{p}_2\left[\frac{\gamma^\mu(\not{p}_1 - \not{k}_1)\gamma^\nu}{2p_1\cdot k_1} + \frac{\gamma^\nu(\not{p}_1 - \not{k}_2)\gamma^\mu}{2p_1\cdot k_2}\right]\right. \quad (5)$$

$$\left.\times\,\not{p}_1\left[\frac{\gamma_\nu(\not{p}_1 - \not{k}_1)\gamma_\mu}{2p_1\cdot k_1} + \frac{\gamma_\mu(\not{p}_1 - \not{k}_2)\gamma_\nu}{2p_1\cdot k_2}\right]\right) \quad (6)$$

Using our rules for traces and gamma matices

$$tr(\not{p}_2\gamma^\mu(\not{p}_1 - \not{k}_1)\gamma^\nu\not{p}_1\gamma_\nu(\not{p}_1 - \not{k}_1)\gamma_\mu) = 32(p_1\cdot k_2)(p_1\cdot k_1) \quad (7)$$

$$tr(\not{p}_2\gamma^\mu(\not{p}_1 - \not{k}_1)\not{p}_1\gamma_\mu(\not{p}_1 - \not{k}_2)\gamma_\nu) = 0 \quad (8)$$

plus two similar terms which just have $k_1 \leftrightarrow k_2$. If we use Mandelstam variables $s = (p_1+p_2)^2 = 2p_1 \cdot p_2, t = (p_1 - k_1)^2 = -2p_1 \cdot k_1, u = (p_1 - k_2)^2 = -2p_1 \cdot k_2$ we can write our answer as

$$< |\mathcal{M}|^2 >= 2g^4 \left[\frac{u}{t} + \frac{t}{u}\right] \tag{9}$$

In c.o.m. frame we have

$$p_1 = (E, 0, 0, E) \qquad p_2 = (E, 0, 0, -E) \tag{10}$$
$$k_1 = (E, E\sin\theta, 0, E\cos\theta) \qquad k_2 = (E, -E\sin\theta, 0, -E\cos\theta) \tag{11}$$

so that

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{4E^2} 2e^2 \left[\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right] \tag{12}$$
$$= \frac{\alpha^2}{s}\left[\frac{1+\cos^2\theta}{1-\cos^2\theta}\right] \tag{13}$$

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We now want to integrate the above formula to get the total cross section. It is straight forward enough to do the ϕ integration giving 2π. To do the θ integration we first note that as the final state particles are photons we limit θ,i.e. $0 \leq \theta \leq \pi/2$. However because we dropped all the dependence on mass the integral is divergent. We can fix this by adding a term proportional to some power of s/m^2 to act as a cutoff. So we get

$$\sigma = \frac{2\pi\alpha^2}{s}\int_0^1 \frac{1+x^2}{1-x^2+p^{-2}}dx \Rightarrow \sigma \sim \frac{2\pi\alpha^2}{s}(\log p) \tag{14}$$

for large c.o.m. energy. So if we choose $p = s/m^2$ we get the expected answer. This is exactly the result we would get if we had kept all the powers of m throughout the calculation. See for example Landau & Lifshitz Quantum Electrodynamics section 88. Now as $\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s}$ we have that

$$R_{\gamma\gamma} = \frac{3}{2}\log\frac{s}{m^2} \tag{15}$$

which for $s \sim (2m_z)^2 = (2\times 92,600)^2 MeV^2$ gives $R_{\gamma\gamma} \sim 36$.

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If we take the transformation given by

$$c \to c' = Uc \tag{16}$$

with the U given in the problem we have that $r \to r' = g, b \to b' = r, g \to g' = b$ and $\bar{c} \to \bar{c}' = \bar{c}U^\dagger$ so $\bar{r} \to \bar{r}' = \bar{g}, \bar{b} \to \bar{b}' = \bar{g}, \bar{g} \to \bar{g}' = \bar{r}$. So

$$|3'\rangle = (r'\bar{r}' - b'\bar{b}')/\sqrt{2} = (g\bar{g} - r\bar{r})/\sqrt{2} \tag{17}$$
$$|8'\rangle = (r'\bar{r}' + b'\bar{b}' - 2g'\bar{g}')/\sqrt{6} = (g\bar{g} + r\bar{r} - 2b\bar{b})\sqrt{6} \tag{18}$$

if we write $|3'\rangle = \alpha\,|3\rangle + \beta\,|8\rangle$ and $|8'\rangle = \gamma\,|3\rangle + \delta\,|8\rangle$ we get

$$\alpha = -1/2 \qquad \beta = -\sqrt{3}/2 \tag{19}$$
$$\gamma = \sqrt{3}/2 \qquad \delta = -1/2 \tag{20}$$

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We can check the realtion $Tr(\lambda^a\lambda^b) = 2\delta^{ab}$ by simple matrix multiplication.

$$\lambda^1\lambda^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda^2\lambda^2 = \lambda^3\lambda^3 \tag{21}$$
$$\lambda^4\lambda^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \lambda^5\lambda^5 \tag{22}$$
$$\lambda^6\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \lambda^7\lambda^7 \tag{23}$$
$$\lambda^8\lambda^8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{24}$$

all of which have the correct trace. For the off diagonal terms and for $a, b \in 1, 2, 3$ we have $tr(\lambda^a\lambda^b) = tr(\sigma^a\sigma^b) = 0$. For

$$\lambda^1\lambda^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$
$$\propto \lambda^1\lambda^5 \tag{26}$$
$$\propto \lambda^2\lambda^4 \tag{27}$$
$$\propto \lambda^2\lambda^5 \tag{28}$$
$$\propto \lambda^3\lambda^6 \tag{29}$$
$$\propto \lambda^3\lambda^7 \tag{30}$$
$$\tag{31}$$

while

$$\lambda^1\lambda^6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$
$$\propto \lambda^1\lambda^7 \tag{33}$$
$$\propto \lambda^2\lambda^6 \tag{34}$$
$$\propto \lambda^2\lambda^7 \tag{35}$$
$$\propto \lambda^3\lambda^4 \tag{36}$$
$$\propto \lambda^3\lambda^5 \tag{37}$$
$$\tag{38}$$

$$
\begin{aligned}
\lambda^4\lambda^6 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (39)\\
&\propto \lambda^4\lambda^7 && (40)\\
&\propto \lambda^5\lambda^6 && (41)\\
&\propto \lambda^5\lambda^7 && (42)\\
&\propto \lambda^3\lambda^4 && (43)\\
&\propto \lambda^3\lambda^5 && (44)\\
& && (45)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^4\lambda^5 &= \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} && (46)\\
\lambda^6\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} && (47)\\
\lambda^1\lambda^8 &\propto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (48)\\
\lambda^2\lambda^8 &\propto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (49)\\
\lambda^3\lambda^8 &\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (50)\\
\lambda^4\lambda^8 &\propto \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (51)\\
\lambda^5\lambda^8 &\propto \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} && (52)\\
\lambda^6\lambda^8 &\propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} && (53)\\
\lambda^7\lambda^8 &\propto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} && (54)\\
& && (55)
\end{aligned}
$$

all of which have obviously zero trace.