Physics 195a
Problem set number 4 – Solution to Problem 18
Due 2 PM, Thursday, October 31, 2002

Notes about course:

- Homework should be turned in to the TA's mail slot on the first floor of East Bridge.

- Collaboration policy: OK to work together in small groups, and to help with each other's understanding. Best to first give problems a good try by yourself. Don't just copy someone else's work – whatever you turn in should be what you think you understand.

- There is a web page for this course, which should be referred to for the most up-to-date information. The URL:
  http://www.hep.caltech.edu/~fcp/ph195/

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- If you think a problem is completely trivial (and hence a waste of your time), you don't have to do it. Just write "trivial" where your solution would go, and you will get credit for it. Of course, this means you are volunteering to help the rest of the class understand it, if they don't find it so simple. . .

READING: Finish reading the "Density Matrix Formalism" course note.

PROBLEMS:

18. Let us try another example of the discussion we have been having in class concerning the use of the uncertainty relation on "localized" wave functions. Consider the three dimensional generalization. Hence, let $P(a)$ be the proability to find the particle, of mass m in a sphere of radius a centered at the origin.

    (a) Recall that in the one dimensional case, if the probability of finding the particle in the interval $(-a, a)$ was α, then a simple lower bound on the kinetic energy was obtained as:

    $$T \geq \frac{1}{8m}\frac{\alpha^2}{a^2}. \tag{1}$$

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Make a simple, but rigorous, generalization of this result to the three dimensional case. Don't worry about finding the "best" bound; even a "conservative" bound may be good enough to answer some questions of interest.

Solution: The limit on T in Eqn. 1 corresponds to a limit on the momentum of

$$\langle p^2 \rangle \geq \frac{1}{4} \frac{\alpha^2}{a^2}. \tag{2}$$

In three dimensions the kinetic energy is

$$T = \frac{1}{2m} \langle p_x^2 + p_y^2 + p_z^2 \rangle. \tag{3}$$

If the probability to find the particle within radius a is $P(a)$, then in each dimension we certainly have that the probability to be in the interval $(-a, a)$ is at least $P(a)$. Thus, we can apply Eqn. 2 in each dimension, and hence, in sum:

$$T \geq \frac{3}{8m} \frac{P(a)^2}{a^2}. \tag{4}$$

(b) We know that an atomic size is of order 10^{-10} m. Suppose that we have an electron which is known to be in a sphere of radius 10^{-10} m with 50% probability. What lower bound can you put on its kinetic energy? Is the result consistent with expectation; *e.g.*, with what you know about the kinetic energy of the electron in hydrogen?

Solution:

$$\begin{aligned} T &\geq \frac{3}{8} \left(\frac{1}{2} \right)^2 \frac{(200 \, \text{MeV-fm})^2}{0.5 \, \text{MeV} 10^{-20} \, \text{m}^2} \\ &\geq 0.8 \text{ eV.} \end{aligned} \tag{5}$$

In the ground state, the expectation value of the kinetic energy of the electron in hydrogen is 13.6 eV, consistent with our bound, although our bound is not especially good.

(c) In ancient times, before the neutron was discovered, it was supposed that the nucleus contained both electrons and protons. A comfortable nuclear size is 5×10^{-15} m. Find a lower bound on

the kinetic energy of an electron if the probability to be within
this radius is 90%. If there is a problem with the validity of your
bound, see if you can fix it.

Solution:

$$
\begin{aligned}
T &\geq \frac{3}{8}(0.9)^2 \frac{(200\,\text{MeV-fm})^2}{0.5\,\text{MeV}25 \times 10^{-30}\,\text{m}^2} \\
&\geq 4\,\text{GeV}.
\end{aligned}
\tag{6}
$$

The electron is relativistic, inconsistent with our assumption in
computing the kinetic energy with a non-relativistic equation. Our
derivation that $\langle p^2 \rangle \geq 1/4a^2$ in the one-dimensional case should
still be valid. The relativistic kinetic energy is $T = E - m \approx \langle |\mathbf{p}| \rangle$,
where the approximation is in the limit $E \gg m$. Let's presume
we can estimate T with $\sqrt{\langle p^2 \rangle}$. Then

$$
\begin{aligned}
T &\geq \frac{\sqrt{3}}{2a} \\
&\geq \frac{200\,\text{MeV-fm}}{5\,\text{fm}} = 40\,\text{MeV}.
\end{aligned}
\tag{7}
$$

Let's compare this with an estimated order of magnitude for the
electrostatic potential energy of an electron and a proton sepa-
rated by 5 fm:

$$
|V| = \frac{e^2}{a} \approx \frac{200\,\text{MeV-fm}}{100 \times 5\,\text{fm}} = 0.4\,\text{MeV}
\tag{8}
$$

This is much smaller than our limit on the kinetic energy, pre-
senting a theoretical problem with binding and with expectations
from the virial theorem.

(d) Now find a lower bound for a proton in the nucleus, if it has a
probability of 90% to be within a region of radius 5×10^{-15} m.

Solution:

$$
\begin{aligned}
T &\geq \frac{3}{8}(0.9)^2 \frac{(200\,\text{MeV-fm})^2}{900\,\text{MeV}25 \times 10^{-30}\,\text{m}^2} \\
&\geq 0.5\,\text{MeV}.
\end{aligned}
\tag{9}
$$

19. Some more thoughts about time reversal: Exercise 13 of the "Ideas of
Quantum Mechanics" course note.

20. The von Neumann mixing theorem: Exercise 3 of the "Density Matrix Formalism" course note.

21. Operators in product spaces: Exercise 4 of the "Density Matrix Formalism" course note.

22. Entropy in a two-state system: Exercise 5 of the "Density Matrix Formalism" course note.

23. Measuring the density matrix: Exercise 6 of the "Density Matrix Formalism" course note.