Physics 195a
Problem set number 6
Due 2 PM, Thursday, November 14, 2002

Notes about course:

- Homework should be turned in to the TA's mail slot on the first floor of East Bridge.
- Collaboration policy: OK to work together in small groups, and to help with each other's understanding. Best to first give problems a good try by yourself. Don't just copy someone else's work – whatever you turn in should be what you think you understand.
- There is a web page for this course, which should be referred to for the most up-to-date information. The URL: http://www.hep.caltech.edu/~fcp/ph195/
- TA: Anura Abeyesinghe, anura@caltech.edu
- If you think a problem is completely trivial (and hence a waste of your time), you don't have to do it. Just write "trivial" where your solution would go, and you will get credit for it. Of course, this means you are volunteering to help the rest of the class understand it, if they don't find it so simple...

READING: Read the "The Simple Harmonic Oscillator: Creation and Destruction Operators" course note.

PROBLEMS:

30. K^0 system in density matrix formalism: Exercise 2 of the K^0 course note.
31. [Worth two problems] "Regeneration": Exercise 3 of the K^0 course note.
32. Qualitative features of wave functions: Exercise 1 of the Harmonic Oscillator course note.

33. One of the failings of classical mechanics is that matter should be "unstable". Let us investigate this in the following system: Consider a system consisting of N particles with masses m_k and charges q_k, $k = 1, 2, \ldots, N$, where we suppose some of the charges are positive and some negative. The Hamiltonian of this multiparticle system is:

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2m_k} + \sum_{N \geq j > k \geq 1} \frac{q_k q_j}{|\mathbf{x}_k - \mathbf{x}_j|},$$

where $p_k = |\mathbf{p}_k|$ is the magnitude of the momentum of the particle labelled "k".

(a) Assume we have solved the equations of the motion, with solutions $\mathbf{x}_k = \mathbf{s}_k(t)$. Show that for any $\omega > 0$ we can select a number $c > 0$ such that $\mathbf{x}_k = c\mathbf{s}_k(\omega t)$ is also a solution of the equations of motion. Remember, we are dealing with the classical equations of motion here.

(b) Find scaling laws relating the total energy, total momentum, total angular momentum, position of an individual particle, and momentum of an individual particle for the original $\mathbf{s}_k(t)$ solution and the scaled $c\mathbf{s}_k(\omega t)$ solution. The only parameter in your scaling laws should be ω. Make sure that any time dependence is clearly stated.

(c) Hence, draw the final conclusion that there does not exist any stable "ground state" of lowest energy. As an aside, what Kepler's law follows from your analysis?

(d) We assert that quantum mechanics does not suffer from this disease, but this must be proven. You have seen (or, if not, see the following problem) the analysis for the hydrogen atom in quantum mechanics, and know that it has a ground state of finite energy. However, it might happen for larger systems that stability is lost in quantum mechanics – there are typically several negative terms in the potential function which could win over the postive kinetic energy terms. We wish to prove that this is, in fact, not the case. The Hamiltonian is as above, but now $\mathbf{p}_k = -i\partial_{\mathbf{k}}$ ($\partial_{\mathbf{k}} = (\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial z_k})$).
Find a rigorous lower bound on the expectation value of H. It doesn't have to be very "good"– any lower bound will settle this

question of principle. You may take it as given that the lower bound exists for the hydrogen atom, since we have already demonstrated this. You may also find it convenient to consider center-of-mass and relative coordinates between particle pairs.

34. The one-electron atom (review?): Continuing from problem 29, now consider the case of the one-electron atom, with an electron under the influence of a Coulomb field due to the nucleus of charge Ze:

$$V(r) = -\frac{Ze^2}{r}, \tag{10}$$

   (a) Without knowing the details of the potential, we may evaluate the form of the radial wave function ($R_{n\ell}(r) = u_{n\ell}(r)/r$, where $\psi_{n\ell m}(\mathbf{x}) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$) for small r, as long as the potential depends on r more slowly than $1/r^2$. Here, n is a quantum number for the radial motion. Likewise, we find the asymptotic form of the wave function for large r, as long as the potential approaches zero as r becomes large. Find the allowable forms for the radial wave functions in these two limits.

   (b) Find the bound state eigenvalues and eigenfunctions of the one-electron atom. [Hint: it is convenient to express the wave function, or rather $u_{n\ell}$, with its asymptotic dependence explicit, so that may be "divided out" in solving the rest of the problem.] You may express your answer in terms of the **Associated Laguerre Polynomials**:

$$L_{n+\ell}^{2\ell+1}(x) = \sum_{k=0}^{n-\ell-1} \frac{(-)^{k+1}(n+\ell)!}{(n-\ell-1-k)!(2\ell+1+k)!k!}x^k. \tag{11}$$