Physics 195a

Problem set number 10 – Solution to Problem 50

Due 2 PM, Thursday, December 12, 2002

READING: Read sections 6 through 9 of the “Angular Momentum” course note.

PROBLEMS:

47. An important lemma in group theory: Exercise 3 of the Angular Momentum course note.

48. The Baker-Campbell-Hausdorff theorem: Exercise 9 of the Angular Momentum course note. This important theorem, or a piece of it, is usually quoted without proof in quantum mechanics texts. This is perhaps not surprising, even if not really excusable – the proof is not trivial. You are given a number of hints which should help to guide you through to the desired result.

49. Application of symmetry principles for selection rules: Exercise 4 of the Angular Momentum course note.

50. Oivil and Livio are up to their old tricks again (sometimes they use their undercover aliases “Alice” and “Bob”, so be leery of anyone bearing these pseudonyms. You have been warned. . .). As usual, they conspire to produce two-photon states. The polarization states of a photon may be described as a two-state system; we'll work in the helicity basis here, with

$$|R\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{90}$$

The Pauli matrix σ_3 is just the helicity operator in this basis, if we adopt the convention that a right-handed particle has positive helicity. If the photon is directed along the positive z-axis, we also have

$$J_z|R\rangle = +|R\rangle, \tag{91}$$

$$J_z|L\rangle = -|L\rangle. \tag{92}$$

The Hermitian matrices σ_1 and σ_2 correspond to other measures of polarization in this state space.

The sneaky Oivil decides to have some fun with his hapless twin. He says: "I want you to measure your photons' polarization in two ways, with operators having eigenvalues of ± 1. Being fair, I'll do the same thing with my photons."

(a) Let $O_1 = \pm 1$ stand for the random variable corresponding to the result of Oivil's measurement with one operator, and O_2 stand for the random variable corresponding to his measurement with his other operator. Likewise, let L_1 and L_2 stand for Livio's random variables. Consider the quantity:

$$T \equiv L_1 O_1 + L_2 O_1 + L_2 O_2 - L_1 O_2. \tag{93}$$

What are the possible values which random variable T can take on?

Let $p(\ell_1, \ell_2, o_1, o_2)$ be the probability of sampling a set of values:

$$L_1 = \ell_1;\ L_2 = \ell_2;\ O_1 = o_1;\ O_2 = o_2. \tag{94}$$

Compute the expectation value, according to $p(\ell_1, \ell_2, o_1, o_2)$ of T:

$$E(T) = E(L_1 O_1 + L_2 O_1 + L_2 O_2 - L_1 O_2), \tag{95}$$

where the $E(T)$ notation is used for "expectation value of T". You don't know $p(\ell_1, \ell_2, o_1, o_2)$, of course, so express your answer in terms of it. Try to find a bound on how large the expectation value can be, independent of $p(\ell_1, \ell_2, o_1, o_2)$. Oivil is a bit too pleased in pointing this bound out to Livio...

Solution: First, let's see what the allowed values of T are:

$$T = (L_1 + L_2) O_1 + (L_2 - L_1) O_2 = \pm 2 \tag{96}$$

The expectation value of T is:

$$\begin{aligned} E(T) &= \sum_{\ell_1,\ell_2,o_1,o_2} p(\ell_1, \ell_2, o_1, o_2)\left[(\ell_1+\ell_2)o_1 + (\ell_2-\ell_1)o_2\right] & (97)\\ &\le 2 \sum_{\ell_1,\ell_2,o_1,o_2} p(\ell_1, \ell_2, o_1, o_2) \\ &\le 2. & (98)\end{aligned}$$

More completely, $-2 \le E(T) \le 2$.

(b) Now Oivil says, "OK, let's try it. We'll produce photons according to the two-photon state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|RR\rangle + |LL\rangle), \tag{99}$$

where L stands for left-handed polarization, and R stands for right-handed polarization. You go measure your photons with the following operators:

$$L_1 = \sigma_3; \quad L_2 = \sigma_1.\text{"} \tag{100}$$

Slyly, he adds, "I'll measure mine with linear combinations of your operators:

$$O_1 = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3); \; O_2 = \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_3).\text{"} \tag{101}$$

Let's see what they might find: What is the expectation value of the operator

$$T = L_1O_1 + L_2O_1 + L_2O_2 - L_1O_2? \tag{102}$$

[Livio was last observed meandering down Venice Beach, shaking his head with a sad demeanor. Do you understand what his problem is, and can you cheer him up?]

Solution:

$$\begin{aligned}
\langle L_1O_1\rangle &= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{3L}(\sigma_{3O} + \sigma_{1O})(|RR\rangle + |LL\rangle) &(103)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{3L}(|RR\rangle - |LL\rangle + |RL\rangle + |LR\rangle)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)(|RR\rangle + |LL\rangle + |RL\rangle - |LR\rangle) &(104)\\
&= \frac{1}{\sqrt{2}}. &(105)\\
\langle L_2O_1\rangle &= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{1L}(\sigma_{3O} + \sigma_{1O})(|RR\rangle + |LL\rangle) &(106)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{1L}(|RR\rangle - |LL\rangle + |RL\rangle + |LR\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)(|LR\rangle - |RL\rangle + |LL\rangle + |RR\rangle) && (107)\\
&= \frac{1}{\sqrt{2}}. && (108)\\
\langle L_2 O_2\rangle &= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{1L}(-\sigma_{3O} + \sigma_{1O})(|RR\rangle + |LL\rangle) && (109)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{1L}(-|RR\rangle + |LL\rangle + |RL\rangle + |LR\rangle)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)(-|LR\rangle + |RL\rangle + |LL\rangle + |RR\rangle) && (110)\\
&= \frac{1}{\sqrt{2}}. && (111)\\
\langle L_1 O_2\rangle &= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{3L}(-\sigma_{3O} + \sigma_{1O})(|RR\rangle + |LL\rangle) && (112)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)\sigma_{3L}(-|RR\rangle + |LL\rangle + |RL\rangle + |LR\rangle)\\
&= \frac{1}{2\sqrt{2}}(\langle RR| + \langle LL|)(-|RR\rangle - |LL\rangle + |RL\rangle - |LR\rangle) && (113)\\
&= -\frac{1}{\sqrt{2}}. && (114)
\end{aligned}
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Therefore,

$$\langle T\rangle = 2\sqrt{2}. \quad (115)$$

Our quantum mechanical result violates the bound we obtained in part (a)!