Physics 195b
Problem set number 11 – Solution to Problem 55
Due 2 PM, Thursday, January 16, 2003

READING: Read sections 10 through 12 of the "Angular Momentum" course note.

PROBLEMS:

51. More on application of symmetry principles for selection rules: Charge Conjugation. Do Exercise 5 of the Angular Momentum course note.

52. Rotational and inversion symmetry, applied to determining selection rules: Exercise 10 of the Angular Momentum course note.

53. Reduction of a representation into irreducible representations: Exercise 11 of the Angular Momentum course note.

54. Little-d functions: Exercise 13 of the Angular Momentum course note.

55. We discussed Berry's phase in class, including the example of the Aharonov-Bohm effect. Now that we know more about rotations, let us try another example.

    Consider a spin-1/2 particle of magnetic moment μ in a magnetic field, $\mathbf{B}$. Let the strength of the magnetic field be a constant, but suppose that its direction is slowly changing. Suppose the magnetic field vector is swept through a closed curve (*i.e.*, think of the tip of its vector as been varied along a closed curve on the surface of a sphere). Thus, two parameters describing the direction of the field are being varied, which we might take to be the polar angles (θ, ϕ).

    (a) What does "slow" mean? That is, on what time scale should the variation be slow, in order for the adiabatic approximation to be reasonable?

    Solution: The Hamiltonian is:

    $$H = -\mu \mathbf{S} \cdot \mathbf{B}(t), \tag{116}$$

    where $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the spin operator. The difference in energy between the ground state (spin along $\mu\mathbf{B}$) and the first excited state (spin opposite $\mu\mathbf{B}$) is just $|\mu\mathbf{B}|$. The rate at which the Hamiltonian changes must be slow compared with this energy gap, for the adiabatic approximation. A suitable measure of the rate of change of the Hamiltonian is

$$\frac{1}{H}\frac{dH}{dt} \sim \frac{1}{|\mathbf{B}|}\left|\frac{d\mathbf{B}}{dt}\right|, \tag{117}$$

where the interpretation of the left hand side should be in terms of expectation values. Thus, for the adiabatic approximation, we require:

$$\frac{1}{|\mathbf{B}|}\left|\frac{d\mathbf{B}}{dt}\right| \ll |\mu\mathbf{B}|. \tag{118}$$

Note that we can also never have the strength of the magnetic field go to zero, since then we would have degenerate ground states.

(b) What is the expectation value of the spin vector in the adiabatic ground state?

Solution: Let θ and ϕ be the polar angles of the magnetic field. These are the paramters which are being varied slowly. Hence, denote the adiabatic ground state by $\psi^{(0)}_{\theta\phi}$. The ground state is the state in which the spin direction is aligned with the magnetic field. The expectation value of the spin vector is therefor:

$$\langle\psi^{(0)}_{\theta\phi}|\mathbf{S}|\psi^{(0)}_{\theta\phi}\rangle = |\mathbf{S}|\frac{\mathbf{B}}{|\mathbf{B}|} = \frac{1}{2}\frac{\mathbf{B}}{|\mathbf{B}|}. \tag{119}$$

(c) Express the adiabatic ground state wave function, $\psi^{(0)}_{\theta\phi}$, in terms of the adiabatically-varied parameters. You can achieve this by inspection from your result to part (b), but I urge you to try doing it instead (or, in addition, as a check) by performing a rotation on a vector to the desired polar angles, now that we know how to do this. I hope you'll even try it with our general rotation matrix formalism (the D matrices), even though you can short-cut this for spin-$\frac{1}{2}$.

Solution: We would like to rotate a vector $\begin{pmatrix}1\\0\end{pmatrix}$, representing a spin along the 3-axis, to a vector oriented with polar angles (θ,ϕ). We may perform such a rotation by first rotating about the 2-axis by angle θ, followed by a rotation about the 3-axis by angle ϕ. We could do this directly, since this is a spin-$\frac{1}{2}$ system, by computing the $SU(2)$ element:

$$u(\theta, \phi) = e^{\phi \mathcal{J}_3} e^{\theta \mathcal{J}_2}, \tag{120}$$

where $\boldsymbol{\mathcal{J}} = -\frac{i}{2}\boldsymbol{\sigma}$. Let us instead use our general rotation formalism, to give it a try. The rotation matrix we desire is (in the Euler angle parameterization):

$$D^{\frac{1}{2}}_{m_1 m_2}(\phi, \theta, 0) = e^{-im_1\phi} d^{\frac{1}{2}}_{m_1 m_2}(\theta). \tag{121}$$

The little-d matrix is given in the notes:

$$d^{\frac{1}{2}}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{122}$$

Thus, the full rotation matrix is:

$$D^{\frac{1}{2}}_{m_1 m_2}(\phi, \theta, 0) = \begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} & -e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\sin\frac{\theta}{2} & e^{\frac{i}{2}\phi}\cos\frac{\theta}{2} \end{pmatrix}. \tag{123}$$

Finally, operating this on our vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we obtain:

$$D^{\frac{1}{2}}_{m_1 m_2}(\phi, \theta, 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-\frac{i}{2}\phi}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}. \tag{124}$$

We may drop the overall phase factor, and write:

$$\psi^{(0)}_{\theta\phi} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}. \tag{125}$$

(d) Suppose we let the **B** field direction rotate slowly around the 3-axis at constant θ. Calculate Berry's phase for this situation. [Recall, from first quarter, that Berry's phase is the change in the phase of the adiabatic wave function (ground state here) over one complete circuit in parameter space. We found that the Berry phase is given by:

$$\gamma_B = i \oint d\boldsymbol{\alpha} \cdot \langle \psi^{(0)}_{\boldsymbol{\alpha}} | \nabla_{\boldsymbol{\alpha}} \psi^{(0)}_{\boldsymbol{\alpha}} \rangle, \tag{126}$$

where $\boldsymbol{\alpha}$ is a vector in parameter space.] See if you can give a geometric interpretation to your answer, in terms of an amount of solid angle swept out by the circuit in parameter space.

Solution: Our parameter space is two-dimensional, but only one is varied along our circuit, so Berry's phase is:

$$\begin{aligned}
\gamma_B &= i\int_0^{2\pi} d\phi \langle \psi^{(0)}_{\theta\phi} | \frac{\partial}{\partial\phi}\psi^{(0)}_{\theta\phi}\rangle && (127)\\
&= i\int_0^{2\pi} d\phi \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}e^{-i\phi}\right)\begin{pmatrix} 0 \\ i\sin\frac{\theta}{2}e^{i\phi}\end{pmatrix} && (128)\\
&= -2\pi\sin^2\frac{\theta}{2}. && (129)
\end{aligned}$$

The geometric interpretation of this result is perhaps not immediately obvious, although the presence of the 2π and the trigonometric function of θ is suggestive. It becomes clearer when we use the identity $\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$. Further, $(1-\cos\theta) = \int_{\cos\theta}^1 d\cos\theta'$ is just the fraction of the total solid angle in parameter space presented by the region encircled by our path. Thus, with $S = \frac{1}{2}$, we may write:

$$\gamma_B = -S\int_0^{2\pi}\int_{\cos\theta}^1 d\cos\theta' = -S\Omega, \tag{130}$$

where Ω is the solid angle encircled by our path, as viewed from the origin in parameter space.