Physics 195b
Problem set number 12 – Solution to Problem 58
Due 2 PM, Thursday, January 23, 2003

READING: Finish reading the "Angular Momentum" course note.

PROBLEMS:

56. Application of $SU(2)$ to nuclear physics: Isospin. Do Exercise 12 of the Angular Momentum course note. This is not a problem on angular momentum, but it demonstates that the group theory we developed for angular momentum may be applied in a formally equivalent context. The problem statement claims that there is an attached picture. This is clearly false. You may find an appropriate level scheme via a google search (you want a level diagram for the nuclear isobars of 6 nucleons), *e.g.*, at:
http://www.tunl.duke.edu/nucldata/figures/06figs/06_is.pdf
For additional reference, you might find it of interest to look up:
F. Ajzenberg-Selove, "Energy Levels of Light Nuclei, A = 5-10," *Nucl. Phys.* **A490** 1-225 (1988)
(see also http://www.tunl.duke.edu/nucldata/fas/88AJ01.shtml).

57. Symmetry and broken symmetry: Application of group theory to level splitting in a lattice with reduced symmetry. Do exercise 14 of the Angular Momentum course note. This is an important problem – it illustrates the power of group theoretic methods in addressing certain questions. I hope you will find it fun to do.

58. In class, we have discussed the transformation between two different types of "helicity bases". In particular, we have considered a system of two particles, with spins j_1 and j_2, in their CM frame.

    One basis is the "spherical helicity basis", with vectors of the form:

$$|j, m, \lambda_1, \lambda_2\rangle, \tag{116}$$

    where j is the total angular momentum, m is the total angular momentum projection along the 3-axis, and λ_1, λ_2 are the helicities of the two particles. We assumed a normalization of these basis vectors such that:

$$\langle j', m', \lambda_1', \lambda_2' | j, m, \lambda_1, \lambda_2\rangle = \delta_{jj'}\delta_{mm'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}. \tag{117}$$

The other basis is the "plane-wave helicity basis", with vectors of the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle, \tag{118}$$

where θ and ϕ are the spherical polar angles of the direction of particle one. We did not specify a normalization for these basis vectors, but an obvious (and conventional) choice is:

$$\langle\theta', \phi', \lambda_1', \lambda_2'|\theta, \phi, \lambda_1, \lambda_2\rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}, \tag{119}$$

where $d^{(2)}\Omega$ refers to the element of solid angle for particle one.

In class, we have obtained the result for the transformation between these bases in the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m} b_j |j, m, \lambda_1, \lambda_2\rangle D^j_{m\delta}(\phi, \theta, -\phi), \tag{120}$$

where $\delta \equiv \lambda_1 - \lambda_2$. Determine the numbers b_j.

Solution: To select a particular b_j, *i.e.*, a particular j, let us invert the basis transformation:

$$\int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi, \theta, -\phi)|\theta, \phi, \lambda_1, \lambda_2\rangle = \tag{121}$$

$$\sum_{j',m'} b_{j'}|j', m', \lambda_1, \lambda_2\rangle \int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi, \theta, -\phi) D^{j'}_{m'\delta}(\phi, \theta, -\phi)$$

$$= \sum_{j',m'} b_{j'}|j', m', \lambda_1, \lambda_2\rangle \int_{4\pi} d\Omega d^j_{m\delta}(\theta) d^{j'}_{m'\delta}(\theta) \exp\left[-i(m'\phi - \delta\phi) + i(m\phi - \delta\phi)\right]$$

$$= \sum_{j',m'} b_{j'}|j', m', \lambda_1, \lambda_2\rangle \int_{-1}^{1} d\cos\theta d^j_{m\delta}(\theta) d^{j'}_{m'\delta}(\theta) \int_0^{2\pi} d\phi e^{i(m-m')\phi} \tag{122}$$

$$= 2\pi \sum_{j'} b_{j'}|j', m, \lambda_1, \lambda_2\rangle \int_{-1}^{1} d\cos\theta d^j_{m\delta}(\theta) d^{j'}_{m\delta}(\theta) \tag{123}$$

$$= 2\pi \sum_{j'} b_{j'}|j', m, \lambda_1, \lambda_2\rangle \frac{2\delta_{jj'}}{2j+1} \tag{124}$$

$$= \frac{4\pi}{2j+1} b_j |j, m, \lambda_1, \lambda_2\rangle. \tag{125}$$

Note that we should probably justify the interchange of the order of summation and integration in the very first step above. Thus,

$$|j, m, \lambda_1, \lambda_2\rangle = \frac{2j+1}{4\pi b_j} \int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi, \theta, -\phi)|\theta, \phi, \lambda_1, \lambda_2\rangle. \tag{126}$$

Now,

$$
\begin{aligned}
1 &= \langle j,m,\lambda_1,\lambda_2|j,m,\lambda_1,\lambda_2\rangle & (127)\\
&= \left[\frac{2j+1}{4\pi|b_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi,\theta,-\phi)\int_{4\pi} d\Omega' D^{j}_{m\delta}(\phi',\theta',-\phi')\langle\theta',\phi',\lambda_1,\lambda_2|\theta,\phi,\lambda_1,\lambda_2\rangle \\
&= \left[\frac{2j+1}{4\pi|b_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi,\theta,-\phi)\int_{4\pi} d\Omega' D^{j}_{m\delta}(\phi',\theta',-\phi')\delta(\cos\theta'-\cos\theta)\delta(\phi'-\phi) \\
&= \left[\frac{2j+1}{4\pi|b_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi,\theta,-\phi)D^{j}_{m\delta}(\phi,\theta,-\phi) & (128)\\
&= 2\pi\left[\frac{2j+1}{4\pi|b_j|}\right]^2 \int_{-1}^{1} d\cos\theta\left[d^{j}_{m\delta}(\theta)\right]^2 & (129)\\
&= \frac{4\pi}{2j+1}\left[\frac{2j+1}{4\pi|b_j|}\right]^2. & (130)
\end{aligned}
$$

Therefore, $|b_j|^2 = (2j+1)/4\pi$, or picking a phase convention,

$$b_j = \sqrt{\frac{4\pi}{2j+1}}. \qquad (131)$$

where we assume that it is all right to interchange the summation and integration. Since each term is non-negative (and each finite), there is no potential for cancellations. Hence, if we find convergence for one ordering of the operations, we will for the other as well.

Note that we have used the result of Eqn. 348 of the notes to obtain:

$$\int_{-1}^{1} d\cos\theta\left[d^{j}_{m\delta}(\theta)\right]^2 = \frac{2}{2j+1}. \qquad (132)$$

59. Clebsch-Gordan coefficients, an alternate practical approach: Exercise 16 of the Angular Momentum course note.

60. Application to angular distribution: Exercise 18 of the Angular Momentum course note. While you may apply the formula we derived in class, I urge you to do this problem by thinking about it "from the beginning" – what should be the angular dependence of the spatial wave function? That is, I hope you will try using some "physical intuition" first, and use the formula as a check if you wish. Note that you

are intended to assume that the frame is the rest frame of the spin-1 particle.