Physics 195b
Problem set number 13 – Solution to Problem 64
Due 2 PM, Thursday, January 30, 2003

READING: Read sections 1-5 of the "Approximate Methods" course note.

PROBLEMS:

61. Exercise 15 of the Angular Momentum course note.

62. Exercise 17 of the Angular Momentum course note.

63. More on decay angular distributions: Do exercise 19 of the Angular Momentum course note.

64. Let us discuss some issues relevant to the proof you gave of Yang's theorem (by the way, the original reference is: C. N. Yang, Physical Review **77** (1950) 242) in exercise 10 of the Angular Momentum course note.

    (a) Let us consider the parity again, and try to establish the connection between how I expected you to do the homework, and an equivalent argument based on the identical boson symmetry.

    In general, the parity of a system of two particles, when their state is an eigenstate of the parity operator, may be expressed in the form:

$$P = \eta_{\text{intrinsic}} \quad \eta_{\text{spatial}},$$

    where $P^2 = 1$, $\eta_{\text{intrinsic}}$ refers to any intrinsic parity due to the interchange of the positions of the particles, and η_{spatial} refers to the effect of parity on the spatial part of the wave function. Given a system of two identical particles, and considering the action of parity on $Y_{L0}(\theta, \phi)$, determine the parity of the system for given orbital angular momentum L.

    For a two-photon system, such as we considered in our discussion of Yang's theorem, explicitly consider boson symmetry (*i.e.*, that the wave function is invariant under interchange of all quantum numbers for two identical bosons) to once again determine the effect of parity of the states:

$$|\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle, \quad |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

Also, give the allowed possibilities for the orbital angular momentum for these states, at least at the level of our current discussion. You may now wish to go back to your original derivation of Yang's theorem, and determine where you implicitly made use of the boson symmetry.

Solution: The action of parity on a system of given orbital angular momentum L is:

$$PY_{L0}(\theta,\phi) = Y_{L0}(\pi-\theta,\pi+\phi) \tag{133}$$
$$= (-)^L Y_{L0}(\theta,\phi), \tag{134}$$

since Y_{L0} does not depend on ϕ, and the effect of the tranformation on θ is determined by the even/odd properties of the Legendre polynomials. Note that it is sufficient to consider Y_{L0}, since if $M \neq 0$, we can always rotate to a basis in which $M = 0$, and $[\mathbf{J}, P] = 0$. Thus, the parity of a system of two identical particles of orbital angular momentum L is $(-)^L$.

The $|\uparrow\uparrow\rangle$ is symmetric under interchange of the spins. Therefore, it must be symmetric under interchange of spatial coordinates in order for the total wave function to be symmetric under interchange of the two photons. Thus, the parity of this state is even. The same argument applies for $|\downarrow\downarrow\rangle$ and $|\uparrow\downarrow + \downarrow\uparrow\rangle$. The $|\uparrow\downarrow - \downarrow\uparrow\rangle$ state is odd under spin interchange, hence must be odd under space interchange; the parity of this state is odd.

The even parity states can have even orbital angular momenta, and the odd parity state can have odd orbital angular momenta.

(b) Continuing our discussion of Yang's theorem, there may be some concern about the total spin angular momentum of the two photon states, and whether the appropriate values are possible to give the right overall angular momentum when combined with even or odd orbital angular momenta. Using a table of Clebsch-Gordan coefficients or otherwise, let us try to alleviate this concern. Thus, decompose our four 2-photon helicity states (with J_z values indicated by $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$, $\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$, where the photons are travelling along the $+$ and $-z$ axis) into states of total spin angular momenta and spin projection along the z-axis: $|S, S_z\rangle$. Hence, show that a $J^P = 0^+$ particle may decay into two photons with relative orbital angular momentum $L = 2$ or 0, and a $J^P = 0^-$ particle may decay into two photons with relative angular momentum $L = 1$.

Solution: We are asked to combine the spins of the two photons to determine the given states in terms of the total spin and its projection along the z axis. Using a table of Clebsch-Gordan coefficients, we find:

$$|\uparrow\uparrow\rangle = |22\rangle \tag{135}$$

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{6}}|20\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{3}}|00\rangle \tag{136}$$

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{6}}|20\rangle - \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{3}}|00\rangle \tag{137}$$

$$|\downarrow\downarrow\rangle = |2-2\rangle. \tag{138}$$

Hence, we also have the desired linear combinations:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{3}}|20\rangle + \sqrt{\frac{2}{3}}|00\rangle \tag{139}$$

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |10\rangle. \tag{140}$$

The $J^P = 0^+$ state, with even parity, can only decay to the even parity state $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$. Note that we must have $L = S$, in order to couple to angular momentum 0. We see that this is possible with $L = 0$, coupling to the $|00\rangle$ spin state, or $L = 2$, coupling to the $|20\rangle$ spin state. The $J^P = 0^-$ state, with odd parity, can only decay to the odd parity state $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. This is possible only with $L = 1$.

65. Do exercise 1 of the Approximate Methods course note.