Physics 195b
Problem set number 16 – Solution to Problem 79
Due 2 PM, Thursday, February 20, 2003

READING: Finish reading the "Scattering" course note.

PROBLEMS:

75. Potential well/bump and spherical solutions: Do Exercise 2 of the Scattering course note.

76. The "fundamental" and "effective" cross sections: Do Exercise 4 of the Scattering course note.

77. Parity conservation and scattering amplitudes: Do Exercise 5 of the Scattering course note.

78. Resonant scattering of light on an atom: Do Exercise 6 of the Scattering course note.

79. When we talked about the hyperfine splitting in atoms, we mentioned that the magnetic dipole moment of the proton is:

$$\boldsymbol{\mu}_p = g_p \frac{e}{2m_p} \mathbf{s}_p, \tag{139}$$

with a measured magnitude corresponding to a value for the gyromagnetic ratio of $g_p = 2 \times (2.792847337 \pm 0.000000029)$. Recall also that I mentioned that the prediction of the Dirac equation for a point spin-1/2 particle is $g = 2$. We may understand the fact that the proton gyromagnetic ratio is not two as being due to its compositeness: In the simple quark model, the proton is made of three quarks, two "ups" (u), and a "down" (d). The quarks are supposed to be point spin-1/2 particles, hence, their gyromagnetic ratios should be $g_u = g_d = 2$ (up to higher order corrections, as in the case of the electron). Let us see whether we can make sense out of the proton magnetic moment.

The proton magnetic moment should be the sum of the magnetic moments of its constituents, and any moments due to their orbital motion in the proton. The proton is the ground state baryon, so we assume that the three quarks are bound together (by the strong interaction) in a state with no orbital angular momentum. By Fermi statistics, the two identical up quarks must have an overall odd wave function under interchange of all quantum numbers. We must apply this with a bit of care, since we are including "color" as one of these quantum numbers here.

Let us look a little at the property of "color". It is the strong interaction analog of electric charge in the electromagnetic interaction. However, instead of one fundamental dimension in charge, there are three color directions, labelled as "red" (r), "blue" (b), and "green" (g). Unitary transformations in this color space, up to overall phases, are described by elements of the group $SU(3)$, the group of unitary unimodular 3×3 matrices. Just like combining spins, we may combine three colors according to a Clebsch-Gordan series, with the result:

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1. \tag{140}$$

We haven't studied this group, so this decomposition into irreducible representations of the product representation is probably new to you. However, the essential aspect here is that there is a singlet in the decomposition. That is, it is possible to combine three colors in such a way as to get a color-singlet state, *i.e.*, a state with no net color charge. These are the states of physical interest for our observed baryons, according to a postulate of the quark model. After some thought (perhaps involving raising and lowering operators along different directions in this color space), you could probably convince yourself that the singlet state in the decomposition above must be antisymmetric under the interchange of any two colors. Assuming this is the case, write down the color portion of the proton wave function.

Now that you know the color wave function of the quarks in the proton, write down the spin wave function.

Since the proton is uud and its isospin partner the neutron is ddu, and $m_p \approx m_n$, let us make the simplfying assumption that $m_u = m_d$. Given the measured value of g_p, what does your model give for m_u? Recall that the up quark has electric charge $2/3$, and the down quark has electric charge $-1/3$, in units of the positron charge.

Finally, use your results to predict the gyromagnetic moment of the neutron, and compare with observation.

Solution: Note that there are six permutations of the three colors among the three quarks, if no two quarks have the same color. The completely antisymmetric combination of three colors is:

$$\frac{1}{\sqrt{6}}(|rgb\rangle - |rbg\rangle + |brg\rangle - |bgr\rangle + |gbr\rangle - |grb\rangle). \tag{141}$$

To construct the spin wave function, we first note that the three spin-1/2 quarks must combine in such a way as to give an overall spin-1/2 for the proton. Second, since the space wave function is symmetric, and the color wave function is antisymmetric, the spin wave function of the two up quarks must be symmetric. Thus, the two up quarks are in a spin 1 state. To give the spin wave function of the proton, let us pick the z axis to be along the spin direction. Then the spin state is:

$$|\frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|11;\frac{1}{2} - \frac{1}{2}\rangle - \frac{1}{\sqrt{3}}|10;\frac{1}{2}\frac{1}{2}\rangle. \tag{142}$$

The magnetic moment of the proton in this model is thus:

$$\mu_p = \frac{2}{3}(2\mu_u - \mu_d) + \frac{1}{3}\mu_d = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d. \tag{143}$$

Hence,

$$g_p \frac{e}{2m_p} = \frac{4}{3}2\frac{2}{3}\frac{e}{2m_u} - \frac{1}{3}2\left(-\frac{1}{3}\right)\frac{e}{2m_d}. \tag{144}$$

With $g_p = 5.58$, $m_p = 938$ MeV, and $m_u = m_d$, we obtain

$$m_u = 2m_p/g_p = 336 \text{ MeV}. \tag{145}$$

The neutron wave function may be obtained from the proton wave function by interchanging the u and d quark labels. Thus,

$$\mu_n = \frac{2}{3}(2\mu_d - \mu_u) + \frac{1}{3}\mu_u = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u. \tag{146}$$

We predict the gyromagnetic moment of the nuetron to be:

$$\frac{\mu_n}{\mu_p} = \frac{\frac{4}{3}\mu_d - \frac{1}{3}\mu_u}{\frac{4}{3}\mu_u - \frac{1}{3}\mu_d} \tag{147}$$

$$= \frac{\frac{4}{3}\left(-\frac{1}{3}\right) - \frac{1}{3}\frac{2}{3}}{\frac{4}{3}\frac{2}{3} - \frac{1}{3}\left(-\frac{1}{3}\right)} \tag{148}$$

$$= -\frac{2}{3}. \tag{149}$$

That is, we predict (neglecting the mass difference) $g_n = -3.72$. This may be compared with the observed value of -3.83.