Physics 195b
Problem set number 18 – Solution to Problems 86 and 87
Due 2 PM, Thursday, March 6, 2003

READING: Read the "Electromagnetic Interactions" course note.

PROBLEMS:

84. Extended boson principle and decays to two pions: Do Exercise 2 of the Identical Particles course note.

85. Gauge transformation in electromagnetism: Do Exercise 1 of the Electromagnetic Interactions course note.

86. We discussed the method of stationary phase in class. Recall that the problem it addresses is to evaluate integrals of the form:

$$I(\epsilon) = \int_{-\infty}^{\infty} f(x) e^{i\theta(x)/\epsilon} dx, \tag{157}$$

where f and θ are real, and $\epsilon > 0$. We showed that, in the situation where ϵ is very small, and θ has a stationary point at $x = x_0$, this integral is approximately:

$$I(\epsilon) = \sqrt{\epsilon} f(x_0) e^{i\theta(x_0)/\epsilon} e^{i\frac{\pi}{4}\mathrm{sign}[\theta''(x_0)]} \sqrt{\frac{2\pi}{|\theta''(x_0)|}} \left[1 + O(\epsilon)\right]. \tag{158}$$

If there is more than one stationary point, then the contributions are to be summed.

To get a little practice applying this method, evaluate the following integral for large t:

$$J(t) = \int_0^1 \cos\left[t(x^3 - x)\right] dx. \tag{159}$$

Solution: To start to get it into the desired form, write

$$J(t) = \Re \int_0^1 e^{i\left(t(x^3-x)\right)} dx. \tag{160}$$

Thus, $f(x) = 1$, $\theta(x) = x^3 - x$, and $\epsilon = 1/t$. The first two derivatives are $\theta'(x) = 3x^2 - 1$ and $\theta''(x) = 6x$. The first derivative is zero at

$x = \pm 1/\sqrt{3}$. The zero at $x_0 = 1/\sqrt{3}$ falls within the range of the integral, so this is the only stationary point of interest. The value of θ at the stationary point is $\theta(x_0) = -2/3\sqrt{3}$. The second derivative at the stationary point is $\theta''(x_0) = 2\sqrt{3}$.

Plugging into our stationary phase formula, we get:

$$J(t) \approx \Re \frac{1}{\sqrt{t}} e^{-\frac{2it}{3\sqrt{3}}} e^{i\pi/4} \sqrt{\frac{2\pi}{2\sqrt{3}}} \tag{161}$$

$$= \Re \sqrt{\frac{\pi}{t\sqrt{3}}} e^{i\left(\frac{\pi}{4} - \frac{2t}{3\sqrt{3}}\right)} \tag{162}$$

$$= \sqrt{\frac{\pi}{t\sqrt{3}}} \cos\left(\frac{\pi}{4} - \frac{2t}{3\sqrt{3}}\right). \tag{163}$$

87. In problem 82 you considered the scattering of particles in a multiplet. You determined the total elastic (sometimes called "scattering") cross section and the total inelastic ("reaction") cross sections in terms of the $A^{(\ell)}_{\alpha\beta}$ matrix in the partial wave expansion. Consider now the graph in Fig. 2.

    This graph purports to show the allowed and forbidden regions for the total elastic and inelastic cross sections in a given partial wave ℓ. Derive the formula for the allowed region of this graph. Make sure to check the extreme points.

    Solution: For simplicity, let the vertical axis be v, and the horizontal axis u:

$$u = \frac{k^2 \sigma^{\text{inel}(\ell)}_{\alpha\text{TOT}}}{\pi(2\ell+1)}; v = \frac{k^2 \sigma^{\text{el}(\ell)}_{\alpha\text{TOT}}}{\pi(2\ell+1)}. \tag{164}$$

From the solution to problem 82, and unitarity of the $A^{(\ell)}$ matrix, we thus have

$$u = \sum_{\beta\neq\alpha} |A^{(\ell)}_{\beta\alpha}|^2 = 1 - |A^{(\ell)}_{\alpha\alpha}|^2, \tag{165}$$

$$v = |A^{(\ell)}_{\alpha\alpha} - 1|^2 = 1 + |A^{(\ell)}_{\alpha\alpha}|^2 - 2\Re A^{(\ell)}_{\alpha\alpha}. \tag{166}$$

The constraint imposed by unitarity is that $|A^{(\ell)}_{\alpha\alpha}|^2 \leq 1$. Let $A^{(\ell)}_{\alpha\alpha} = re^{i\theta}$. Then $r \leq 1$ and $0 \leq \theta < 2\pi$ gives the allowed region. In terms of the plotted quantities, $u = 1 - r^2$ and $v = 1 + r^2 - 2r\cos\theta$. Thus

$$0 \leq u \leq 1, \tag{167}$$

and for given u, v must be in the range

$$(1-r)^2 \leq y \leq (1+r)^2, \tag{168}$$

where $r = \sqrt{1-x}$. If $r = 0$ then $(u, v) = (1, 1)$. If $r = 1$ then $u = 0$ and $0 \leq v \leq 4$.

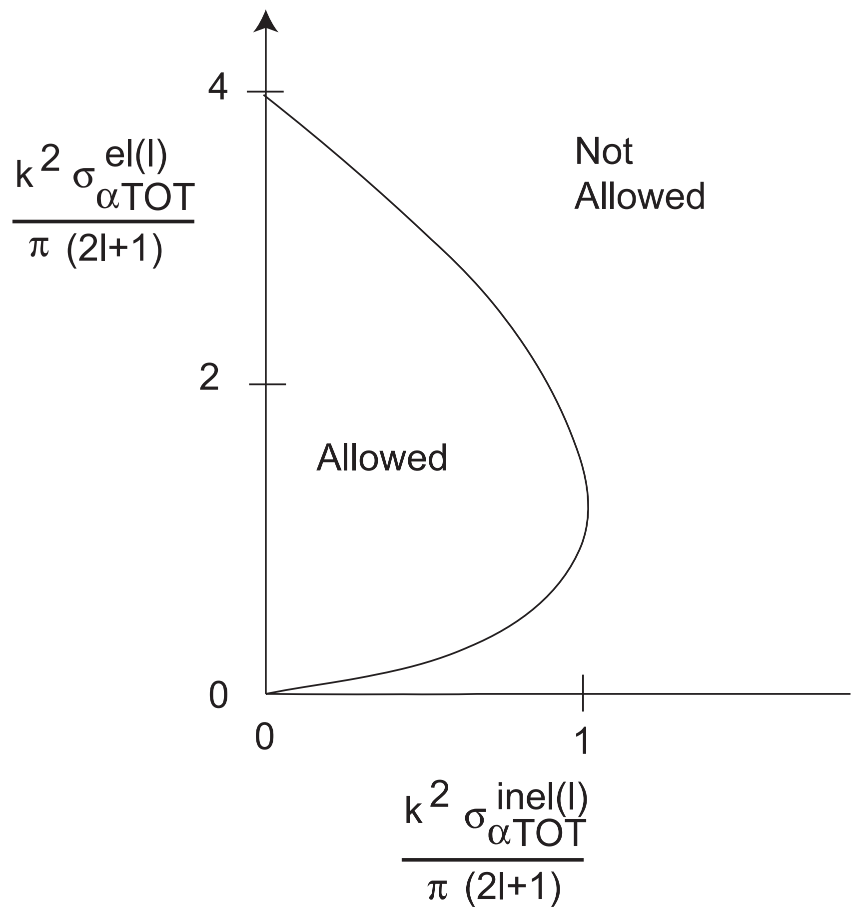

Figure 2: The allowed and forbidden regions for possible elastic and inelastic cross sections for the scattering of particles in a multiplet.