Physics 195b
Problem set number 20
Due 2 PM, Thursday, March 20, 2003

Notes about course:

- Homework should be turned in to the TA's mail slot on the first floor of East Bridge.
- Collaboration policy: OK to work together in small groups, and to help with each other's understanding. Best to first give problems a good try by yourself. Don't just copy someone else's work – whatever you turn in should be what you think you understand.
- There is a web page for this course, which should be referred to for the most up-to-date information. The URL: http://www.hep.caltech.edu/~fcp/ph195/
- TA: Anura Abeyesinghe, anura@caltech.edu
- If you think a problem is completely trivial (and hence a waste of your time), you don't have to do it. Just write "trivial" where your solution would go, and you will get credit for it. Of course, this means you are volunteering to help the rest of the class understand it, if they don't find it so simple...

PROBLEMS:

92. We obtained the following Hamiltonian, up to a constant, for our superconductor:

$$\begin{aligned} H_S &= \int d^3(\mathbf{x}) \left[\frac{1}{2m}\nabla\psi_s^\dagger(\mathbf{x})\cdot\nabla\psi_s(\mathbf{x}) - \mu\psi_s^\dagger(\mathbf{x})\psi_s(\mathbf{x})\right] \\ &+\frac{1}{2}\int d^3(\mathbf{x})\int d^3(\mathbf{y})\left[\Delta_{ss'}(\mathbf{x},\mathbf{y})\psi_{s'}(\mathbf{y})\psi_s(\mathbf{x}) + \Delta^*_{ss'}(\mathbf{x},\mathbf{y})\psi_s^\dagger(\mathbf{x})\psi_{s'}^\dagger(\mathbf{y})\right], \end{aligned} \tag{169}$$

where

(a) $\psi_s^\dagger(\mathbf{x})$ is a field operator creating an electron with spin projection s at point $\mathbf{x}$,

(b) repeated indices are summed over,

(c) μ is the "chemical potential" (energy of the Fermi surface here),

(d) the "gap function" is:

$$\Delta_{ss'}(\mathbf{x},\mathbf{y}) = \langle \psi_s^\dagger(\mathbf{x})\psi_{s'}^\dagger(\mathbf{y})\rangle U(\mathbf{x}-\mathbf{y}), \tag{170}$$

(e) and $U(\mathbf{x}-\mathbf{y})$ is the (weakly attractive) electron-electron potential.

As a first step towards the Bogoliubov transformation, we defined the two-component vector field according to:

$$\Upsilon_\alpha = \begin{pmatrix} \psi_s(\mathbf{x}) \\ \psi_s^\dagger(\mathbf{x}) \end{pmatrix}, \tag{171}$$

where α subsumes the s and $\mathbf{x}$ indices in one symbol.

Show that our Hamiltonian may be written in the form:

$$H_S = \frac{1}{2}\Upsilon_\alpha^\dagger H_{\alpha\beta}\Upsilon_\beta + H_0, \tag{172}$$

where

$$H_0 = \frac{1}{2}\mathrm{Tr}\left(-\frac{1}{2m}\nabla^2 - \mu\right), \tag{173}$$

and

$$H_{\alpha\beta} = \begin{pmatrix} \left(\frac{1}{2m}\nabla_x\cdot\nabla_y - \mu\right)\delta_{\alpha\beta} & \Delta^*_{\alpha\beta} \\ -\Delta_{\alpha\beta} & -\left(\frac{1}{2m}\nabla_x\cdot\nabla_y - \mu\right)\delta_{\alpha\beta} \end{pmatrix}. \tag{174}$$

93. Let us investigate the very low energy scattering limit somewhat further. In this limit, we expect S-wave scattering to dominate, so let us look at the S-wave term (considering spinless case for now):

$$f_0(k) = \frac{1}{2ik}\left(e^{2i\delta_0} - 1\right). \tag{175}$$

In the low energy limit, we can expand $k\cot\delta_0$ in a series in powers of k^2:

$$k\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0k^2 + O(k^4), \tag{176}$$

where a is called the (zero-energy) "scattering length",[1] and r_0 is called the "effective range". In principle, we need to establish this formula, and relate a and r_0 to the properties of the scattering center. However, let us assume that the energy is sufficiently low that we may neglect all but the first term.

(a) Show that, in the low energy limit, we may write $\delta_0 = -ak$, and find a simple "physical" picture for a. That is, I want you to relate a to some value of radius r. It is possible to do this by considering the wavefunctions only in the region where the potential vanishes, although to actually compute a requires looking at the potential, of course. What is the total cross section, in terms of the scattering length?

(b) We continue by considering specifically very low energy neutron-proton scattering. These are not spinless particles, and indeed it is observed that the potential is spin-dependent (can you think why you already know this?). Thus, for neutron-proton scattering at very low energies, we have two potentials to think about (more, if spin flips happen, but we assume that we are at sufficiently low energy so that everything is S-wave, with total spin conserved): An effective potential for scattering in the spin-singlet state, and another for the spin-triplet state. Corresponding to these two potentials, we introduce two scattering lengths, a_t for the triplet interaction, and a_s for the singlet.

With the sign convention of part (a), and using what you know about the neutron-proton interaction, give a simple discussion of what you expect for the signs of a_t and a_s.

(c) Continuing with low energy neutron-proton scattering, show that the total cross section for low energy scattering of neutrons on a target of randomly polarized protons is:

$$\sigma_0 = 4\pi\left(\frac{3}{4}a_t^2 + \frac{1}{4}a_s^2\right). \tag{177}$$

94. In practice, the result of the preceding problem may be applied to neutron scattering on a hydrogen target (as our source of protons), when the neutron wavelength is short compared with the hydrogen

[1] Not everyone uses the same sign convention for a!

molecular size, but still long enough to be in the "low-energy" regime, where we may neglect both higher partial waves and the effective range term.

(a) Consider a possible target with hydrogen gas at room temperature. Assuming thermal equilibrium, such a gas is a mixture of parahydrogen (nuclear spins antiparallel) and orthohydrogen (nuclear spins parallel).[2] What is the fraction of parahydrogen?

(b) Suppose now that we have constructed a target (at 20 K, say) consisting entirely of parahydrogen. We consider the scattering of "cold" neutrons from this target. By "cold" we mean neutrons with an energy corresponding also to $T \approx 20$ K. In this case (as you should convince yourself with a quick computation), the neutrons scatter elastically from the parahydrogen molecule as a whole. This may be thought of as coherent scattering from the two protons. Show that the total cross section for the scattering of cold neutrons on parahydrogen is:

$$\sigma_P = 4\pi\frac{64}{9}\left(\frac{3}{4}a_t + \frac{1}{4}a_s\right)^2. \tag{178}$$

You should assume: (*i*) The neutron wavelength is much larger than the molecular size; (*ii*) The scattered wave is the sum of the waves scattered from each proton – *i.e.*, there is no "double scattering" where the wave scattered from one proton subsequently scatters on the second proton; (*iii*) The scattering is strictly elastic.

Experimentally (see *e.g.*, R. B. Sutton *et al.*, Phys. Rev. **72** (1947) 1147),

$$\sigma_0 = 20.4 \times 10^{-24}\,\text{cm}^2, \tag{179}$$
$$\sigma_P = 3.9 \times 10^{-24}\,\text{cm}^2. \tag{180}$$

Thus, determine a_t and a_s. Are your results sensible?

[2] How do you remember which is which? Life is complicated by paradoxical parallelisms, to say nothing of orthogonal orthodoxisms. But in the end, orthodoxy is paramount.