Physics 231a
Problem Set Number 1
Due Wednesday, October 6, 2004

Note: Some problems may be "review" for some of you. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

Overview

Particle physics is concerned with the structure of matter and its interactions at the most basic level.

The term "High Energy Physics" is often used as a synonym for particle physics. The reason for this is that probing matter at this level is often done using particle beams which have been accelerated to "high" energies. The scale for "high" is set by the distance scale which one is attempting to probe. According to the deBroglie relation, the higher the energy, the smaller the wavelength of the probe. For example, to have a probe with a wavelength of order nuclear size, it needs to have a momentum:

$$p = 1/\lambda\!\!\!{}^{-} \sim 1/(1\,\mathrm{fm}) \sim 200\,\mathrm{MeV}. \qquad (1)$$

1 The Standard Model

The current situation is that there is a theoretical description of the strong and electroweak interactions, called the standard model, which, together with general relativity, forms a description of Nature passing all existing tests, presuming we incorporate recent evidence for "neutrino oscillations". The standard model is a relativistic quantum field theory describing the particle content and the interactions among the particles. There are good reasons to believe that the standard model must break down in some way, and is ultimately only an approximate description of Nature. For example, a quantum mechanical treatment of gravity is not incorporated. The standard model also leaves a number of questions unanswered, such as the mass hierarchy of the generations, which is simply put into the theory "by hand". Unless otherwise stated, the standard model will be assumed in the following discussion.

The particle content of the standard model consists of fundamental (point-like) spin-1/2 fermions, spin-1 gauge bosons, and a spin-0 Higgs particle. The fermions come in three generations of strongly interacting quarks and non-stongly interacting leptons:

$$\begin{pmatrix} u_r & u_g & u_b \\ d_r & d_g & d_b \end{pmatrix}, \begin{pmatrix} c_r & c_g & c_b \\ s_r & s_g & s_b \end{pmatrix}, \begin{pmatrix} t_r & t_g & t_b \\ b_r & b_g & b_b \end{pmatrix}, \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}. \quad (2)$$

The "r", "g", "b" subscripts refer to the quark "color" states, "red", "green", and "blue". The generations differ only in the masses – otherwise their interactions are identical, *i.e.*, a muon is just like an electron, except that it is about 200 times heavier. All of these fermions have corresponding anti-particles, with opposite charges (color, electric, and weak).

The gauge bosons are the photon, γ, mediating the electromagnetic interaction, the three weak bosons, $W^\pm, Z$, and the eight gluons, g_i, $i = 1 \ldots 8$, carrying the strong force. The photon and the gluons are massless. As the photon also carries zero electric charge, the electromagnetic force is "long range", *i.e.*, Coulombic. The gluons, however, are themselves strongly charged, leading to a picture where the strong force is approximately Coulombic at short distances ("asymptotic freedom"), but becomes strong at large distances. This results in the notion of "confinement", in which the physical hadrons are color singlets.

The weak bosons are not massless, in fact they are two orders of magnitude heavier than the nucleon. Giving a gauge boson a mass in a gauge field theory must be done carefully. Simply putting in a mass term destroys gauge invariance, making the theory "unrenormalizable", which therefore loses all predictive power in perturbation theory. The way around this is to preserve the underlying gauge invariance, but to appear to break it through a choice of initial condition. The idea is to add a term to the Lagrangian with a gauge-symmetric, but degenerate ground state. The gauge symmetry is apparently broken ("spontaneous symmetry breaking") by choosing a particular ground state to expand about. The degrees of freedom employed in the choice of ground state become longitudinal polarization degrees of freedom of the gauge particles – *i.e.*, the gauge particles acquire mass. This is referred to as the "Higgs mechanism".

A consequence of this procedure is that one or more new physical particles appear. In the standard model, only one such particle appears, a neutral scalar particle called the "Higgs boson". The Higgs boson has not yet been experimentally observed, and this aspect remains one of the important poorly tested areas of the standard model. Extensions to the standard model sometimes require more than one Higgs particle.

The weak interaction is further complicated in two ways. One of these is that the coupling to fermions is observed to only be to left-handed fermions, and right-handed antifermions, for example, as observed in nuclear beta decay. This is incorporated in the standard model by explicitly putting only left-handed couplings into the weak interaction Lagrangian. To preserve gauge invariance, the fermions must also acquire their masses via coupling with the Higgs field. The masses are proportional to the coupling strength, conversely, the Higgs coupling is stronger to heavier quarks and leptons.

The second complication is that the neutral interactions underlying the left-handed coupling ($SU(2)_L$) and the left-right symmetric coupling ($U(1)_Y$, where Y, "hypercharge", refers to the charge in this gauge group) are mixed to give the physical electromagnetic (photon) and weak (Z) neutral gauge bosons. The photon is fixed to be massless in the standard model, and the Z mass is shifted by the mixing to a higher value than its charged weak partners. The photon retains left-right symmetric couplings, but the Z acquires a right-handed coupling in addition to the purely left-handed $SU(2)_L$ coupling. Because of this mixing, the weak and electromagnetic interactions are said to be "unified" into the "electroweak" interaction, though no reduction in parameters is achieved.

2 Beyond the Standard Model

Successful though it is, the standard model is not expected to be the final word. There are several reasons why people think this, for example:

- Gravity is not given a quantum mechanical treatment.
- The standard model treatment of CP violation, as coming from the quark mixing matrix, is apparently insufficient to explain the magnitude of the baryon-antibaryon asymmetry in the universe, at least in the "standard" cosmology.
- A complete understanding might be expected to predict the values of all constants, other than arbitrary scales. With over 20 input parameters to be fixed by experiment, the standard model does not satisfy this criterion.

- There is a “fine-tuning” problem in the strong interaction, in which there is an arbitrary parameter which must be extremely close to zero in order to avoid strong CP-violation in disagreement with experiment. The standard model provides no reason why this parameter should be close to zero. A simple extension can provide this reason, at the expense of introducing a new particle, the “axion”. However, the minimal way of making this extension yields predictions in disagreement with experiment, so additional extensions are required to make it work.

There are potential extensions of the standard model which address some or all of these concerns. For prominent examples:

2.1 GUTs

Grand Unified Theories, or “GUTs”, are motivated by the desire to explain the electroweak and strong interactions as low-energy manifestations of a single, unified, interaction. This unified interaction is also a gauge theory, but at a much higher energy scale, perhaps 10^{15} GeV. The symmetry is apparently broken in the low-energy regime, to give the standard model.

There are various possible GUTs, with different gauge groups. A typical feature in these theories is the prediction of proton decay, or non-conservation of baryon number. The present experimental limits on proton decay are sufficient to rule out many of these models.

2.2 Supersymmetry

Supersymmetry refers to a symmetry between bosons and fermions. Thus, to each fermion in the standard model a partner boson is added, and to each boson, a partner fermion. The symmetry is broken, in that the mass spectra of the particles and the supersymmetric partners (“sparticles”) are different. The fundamental coupling strengths are, however, the same.

2.3 Superstrings

Superstring theories are candidates for the “theory of everything” (“TOE”). Gravity is included, and supersymmetry is probably required. The “string” appelation refers to the fact that the fundamental objects are no longer point-like, but have a one-dimensional characteristic.

2.4 Compositeness

Compositeness is a generic idea that one or more of the particles which are point-like in the standard model, are actually a composite of other particles. To agree with experiment, the compositeness scale must generally be smaller than of order:

$$1/1\,\text{TeV} \sim 10^{-19}\,\text{m}. \tag{3}$$

2.5 And there are more...

3 Experimental Techniques

Fundamental particles and their interactions are experimentally accessible when very small distance scales may be probed. There are two basic approaches to doing this:

Conceptually the simplest is to use a high-power "microscope". This means using very short-wavelength probes to resolve the distance scales of interest. Such probes are manufactured using particle accelerators. They are also available, though uncontrollable, in astrophysical sources, including the early universe.

The other major experimental technique is to use rare processes which may be accessible at lower energies. For example, while a physical hadron is a rather large object, the wave function may be such that the constituents have a small but finite probability of getting very close to each other. By looking for processes which occur in this rare circumstance, it is possible to study small distance scales.

Many of the realizations of either method employ man-made particle accelerators. The results of collisions or decay processes are measured in particle detectors, often themselves very large facilities in modern experiments.

Conventions and Kinematics

There are several conventions, such as the system of units, used in particle physics which are not so commonly used in other fields. There are also conventions which are not universal even within particle physics.

An important tool in particle physics is kinematics, describing the results of interaction processes. Because of the relativistic nature of many interesting processes, relativistic kinematics is generally required.

4 Spacetime

In Minkowski spacetime, the 0-component is taken to be the time coordinate, and the metric signature $(1, -1, -1, -1)$ is adopted here. Neither of these are universal conventions, and it is sometimes important to know which convention is being used. For example, in the present convention, the completely antisymmetric tensor in four dimensions is

$$\epsilon_{abcd} = \delta(abcd), \tag{4}$$

where $\delta(abcd)$ is the parity of the permutation of the four numbers a, b, c, d, such that $\delta = 0$ if any two of the numbers are equal. The same physical ordering of indices gives the opposite sign, if the 4-component is taken to be the time coordinate.

5 Units

A convenient system of units to use in particle physics is the "rationalized Heavyside-Lorentz" system. In this system, $\hbar = c = 1$, and charge is measured in units such that Maxwell's equations appear in the form:

$$\nabla \cdot D = \rho \tag{5}$$
$$\nabla \times E = -\partial_t B \tag{6}$$
$$\nabla \cdot B = 0 \tag{7}$$
$$\nabla \times H = \partial_t D + J \tag{8}$$

The Lorentz force law retains its familiar form:

$$F = q(E + v \times B). \tag{9}$$

It is important to realize that in these units, the fine structure constant is

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}. \tag{10}$$

The reader is advised to verify this.

A commonly used length unit is the femtometer (or "fermi"), fm, equal to 10^{-15} m. Time may be measured directly in fm, or in seconds, with conversion constant[1] $1 = 2.99792458 \times 10^8\,\mathrm{m/s}$. Alternatively, distances are sometimes measured in seconds. For example, in accelerators we may measure distances in terms of the time it takes a speed 1 particle to traverse the distance: "The BES e^+e^- storage ring has a circumference of 800 ns."

Particle lifetimes are quoted in terms of the mean decay time for a particle at rest. For a particle described as a Breit Wigner resonance,

$$\frac{dn}{dm} = \frac{N}{(m - m_0)^2 + \Gamma^2/4}, \tag{11}$$

the lifetime is $1/\Gamma$.

Another very handy conversion constant is

$$1 = \hbar c \approx 197.3\,\mathrm{MeV\text{-}fm}. \tag{12}$$

6 Notation

The same symbols will usually be used to represent an operator as for values in its spectrum. In cases where the distinction is useful, a "hat" ($\hat{}$) will be placed over the symbol to denote the operator. The distinction between scalars, three-vectors, and four-vectors will sometimes be assumed by context, rather than by specific notation, especially where the distinction isn't really important. Where the context may not be sufficient, three-vectors are denoted with bold-face font.

7 (Lack of) Rigor

The emphasis is on the phenomenology, and on the ideas. Typically, subtle questions of rigor are not dealt with. Issues of domain are generally not checked in operator equations. Completeness of systems of states is usually assumed without verification.

[1] This is an exact number, providing the definition of the meter.

8 Kinematics

An element of the Poincare, or inhomogeneous Lorentz group, acting in the space of 4-vectors, may be represented by a 5×5 matrix:

$$\Lambda = \begin{pmatrix} & & & & z_0 \\ & & & & z_1 \\ & & \Lambda_0 & & z_2 \\ & & & & z_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

where the 4-vector is represented in the form:

$$v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}. \tag{14}$$

The quantities (z_0, z_1, z_2, z_3) correspond to a translation, while the 4×4 matrix Λ_0 represents a homogeneous Lorentz transformation. To be an element of the homogeneous Lorentz group, Λ_0 must "preserve the invariant interval":

$$(\Lambda_0 v)^2 = v^2, \tag{15}$$

for all 4-vectors v. This constraint implies that all such elements can be represented as products of pure proper spatial rotations times pure (special relativistic) velocity boosts, times, possibly, spatial inversion and/or time inversion. A proper homogeneous Lorentz transformation is one in which neither space inversion nor time inversion is present.

8.1 Special Kinematical Variables

In addition to the momentum and energy, there are several more specialized kinematical variables which are handy in describing particle interactions. In the following, we suppose z is a "special" chosen spatial direction. In practice, it is usually useful to pick this direction to be along the direction of one of the incoming particles in a colinear two-body collision process, or sub-process. In such a case, we refer to this direction as the "longitudinal" direction, and the perpendicular directions as "transverse".

- Rapidity: The rapidity, y, of a particle is an alternative to p_z as a measure of the particle's longitudinal motion. It is defined as:

$$y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right). \tag{16}$$

- Pseudorapidity: The pseudorapidity, η, of a particle is a convenient approximation to the rapidity, valid at high energies and transverse momenta not too near the maximum. It is defined according to:

$$\eta = -\ln \tan \theta/2, \tag{17}$$

  where θ is the angle between the particle and the z-axis.

- Mandelstam Invariants: In a scattering process $a + b \to c + d$, denote the four-momenta of the particles a, b, c, d by p_a, p_b, p_c, p_d, respectively. Define the following three Lorentz scalars:

$$\begin{array}{rclcl} s &=& (p_a + p_b)^2 &=& (p_c + p_d)^2 \\ t &=& (p_c - p_a)^2 &=& (p_d - p_b)^2 \\ u &=& (p_d - p_a)^2 &=& (p_c - p_b)^2 \end{array} \tag{18}$$

  Variable s is the square of the center-of-mass energy of the colliding particles, and t and u may be thought of as squared momentum transfers in the collision. Note that, if there are no spin polarizations, two invariants (besides the particle masses) are sufficient to completely describe the kinematics of such a $2 \to 2$ process. Hence, the three Mandelstam variables are not independent.

- Feynman x: The Feynman x of a particle is similar to the rapidity in that it is a measure of the particle's momentum along some "longitudinal" direction. With the z-axis chosen along this longitudinal direction, it is defined as:

$$x = p_z / p_{z,\max}, \tag{19}$$

  where $p_{z,\max}$ is the maximum longitudinal momentum the particle can have. This variable is typically most useful in the center-of-mass of the collision process (or sub-process, in which case this variable is often called z), and at high enough energies that the masses may be neglected, $x \sim 2p_z/\sqrt{s}$.

- Q^2, ν, x, y, W: There are several commonly-used kinematic variables used in describing deep-inelastic collisions. Consider muon neutrino scattering on a nuclear target, resulting in an outgoing muon (charged current interaction), and hadronic stuff. The "energy transfer", ν, is defined as the difference in energy between the incoming and the outgoing leptons:

  $$\nu = E_\nu - E_\mu. \tag{20}$$

  The "momentum transfer" (squared), Q^2, is the scalar magnitude of the 4-momentum transferred from the leptons to the hadrons (squared):

  $$Q^2 = -(p_\nu - p_\mu)^2. \tag{21}$$

  If M is the mass of the initial hadronic state, the Bjorken scaling variable, x, is defined according to:

  $$x = Q^2/2M\nu. \tag{22}$$

  This is not the same as the Feynman x variable, though there exists a connection in the interpretation, which will be elucidated later. The variable y is defined as the fractional energy transfer from the lepton:

  $$y = \nu/E_\nu. \tag{23}$$

  This is not a rapidity, in spite of the use of the same symbol. Note that most of these quantities are not Lorentz-invariant – they are to be evaluated in the lab frame, where the target is at rest.

Exercises

1. Some kinematics: A high intensity green laser pulse is focussed to make a head-on collision with a 50 GeV electron beam. Suppose that a photon scatters from an electron. What is the minimum number of laser photons required to collide ("simultaneously") with a scattered photon in order to produce an e^+e^- pair?

2. Rapidity variable:

(a) Show that rapidity differences are invariant with respect to longitudinal Lorentz transformations. Therefore, all longitudinal velocity boosts are pure translations in rapidity, with the value of the translation dependent only on the boost velocity.

(b) Show that, in the appropriate regime, $\eta \sim y$. Give the appropriate kinematic region.

3. Show that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.

4. The η meson has spin 0. Thirty-nine per cent of the time it decays to two photons. The mass of the eta is 547 MeV. Imagine doing an experiment where we observe $\eta \to \gamma\gamma$ decays, in which we measure the energies of the decay photons. You may assume a "perfect detector", covering 4π solid angle with infinitely good resolution at all relevant photon energies. Imagine that we measure a large number, N, of η decays.

   (a) If the η is produced exactly at rest (in the detector frame), what is the angular distribution, $\frac{dN}{d\Omega}$, of the observed photons? What is the energy distribution, $\frac{dN}{dE_\gamma}$?

   (b) If the η is moving with momentum p_η, what is the energy distribution $\frac{dN}{dE_\gamma}$?

5. In designing beam lines for charged particles, a useful technique is the use of matrices to describe the effect of beam elements (including a simple drift space) on the trajectory. Let us see how this can be done, using the example of a bend: Consider a dipole magnet with uniform field $\mathbf{B} = B\hat{e}_y$ and length L. Suppose that a beam particle is incident on the magnet gap in the $x - z$ plane. For simplicity, we here ignore motion in the y direction, though it is not difficult to add. Let the particle momentum be $p + \Delta p$, where p is some nominal momentum, and Δp represents a small correction to obtain the actual momentum.

   The x and z coordinates are defined as follows: Consider a beam particle of momentum p, directed such that it is aligned with the length of the magnet at the entrance to the field. It is going in the z direction at the entrance to the magnet, and the x direction is then perpendicular to z, and to $\mathbf{B}$ (right-handed coordinate system). As the particle traverses the magnet, the coordinate system rotates such that the z axis is always along the particle direction.

Describe a beam particle trajectory according to the vector:

$$\mathbf{V} = \begin{pmatrix} x \\ \frac{dx}{dz} \\ \frac{\Delta p}{p} \end{pmatrix}$$

Determine the transfer matrix of the bend magnet, *i.e.*, the matrix which you multiply the initial vector (at the entrance to the magnet) by to get the final vector (at the magnet exit). You may assume that the angles are all small. Good units to use are Tesla, meters, and GeV. Use your matrix representation to determine how far a proton is offset (exit – entrance) transversely by a magnet of strength 1 Tesla, length 1 meter, $p = 10$ GeV, and with an incident angle of 1 mradian with respect to the magnet axis, *i.e.*, $dx/dz(\text{initial}) = 0.001$

6. When we asked why $\rho^0 \to \pi^0\pi^0$ doesn't happen, someone suggested that it might have something to do with flavors. Well, it isn't permitted by angular momentum conservation plus Bose symmetry, but let's pursue this idea a bit further anyway. The strong interaction conserves nuclear isospin. Show that $\rho^0 \to \pi^0\pi^0$ is forbidden to occur strongly even if the angular momentum argument didn't apply (*e.g.*, imagine the spins are something different from the true values).