Physics 231a
Problem Set Number 2
Due Wednesday, October 13, 2004

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

7. Standard Model, $U(1)$ gauge theory: [This will be the first of a series of problems that we will have towards building the entire standard model Lagrangian. I am assuming that this problem is really review, but I would be interested to know if that is not the case.] Suppose there is a Dirac particle of mass m described by field $\psi(x)$, where x is any spacetime point. The Lagrangian for the free particle is:

$$\mathcal{L} = -m\bar{\psi}\psi + i\bar{\psi}\partial\!\!\!/\psi. \tag{24}$$

This Lagrangian yields the Dirac equation when substituted into the Euler-Lagrange equations, where the fields are the generalized coordinates. If you have never demonstrated this, please do so now! The notation $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$ is the Feynman "slash" shorthand, and of course repeated indices are summed over.

We further suppose that we want to build a gauge theory with gauge group $U(1)$, for example, to describe electromagnetism. The gauge transformation on the fermion fields is

$$\psi(x) \to \psi'(x) = e^{i\theta(x)}\psi(x), \tag{25}$$

where $\theta(x)$ is a real function of x. This is a $U(1)$ transformation since $U(1) = \left\{e^{i\theta}\right\}$. We require local gauge invariance, i.e., that the Lagrangian be invariant under such local gauge transformations.

The free particle Lagrangian in Eqn. 25 does not satisfy local gauge invariance. Show that gauge invariance may be obtained by replacing the derivative operator ∂_μ with a "covariant" derivative D_μ of the form:

$$D_\mu \equiv \partial_\mu + igA_\mu(x), \tag{26}$$

and determine how A_μ must transform under a gauge transformation.

In electromagnetism (QED), A_μ is interpreted as the photon field operator, and

$$g = Qe \tag{27}$$

is the coupling strength, where Q is the charge operator on the fermion field in units of the proton charge (i.e., $Q\psi = -\psi$ for the electron field).

You should make sure that this discussion, including your answer, makes sense both in terms of the Schrödinger description of the interacton of a non-relativistic charged particle with an electromagnetic field, and with the notion of gauge invariance in classical electromagnetism.

8. More isospin: The $K^*(892)$ resonance has isotopic spin $\frac{1}{2}$, as does the K meson (*e.g.*, the K^+ meson is made from a u and an $\bar{s}$ and the K^0 is made from a d and an $\bar{s}$). Determine the relative branching ratios:

$$\frac{B(K^{*+}(892) \to K^+\pi^0)}{B(K^{*+}(892) \to K^0\pi^+)}$$

   Remember that isospin is completely analgous to spin. Be sure to discuss any assumptions or approximations you might be making.

9. Some additional useful kinematics: We are often interested in the 2-body decays of a particle of mass M, moving in the lab frame with momentum p. An experimental concern (because of its relevance to detection efficiency) is the angle between the decay products in the lab frame, called the "opening angle". For example, suppose our particle decays to two photons.

   (a) What is the minimum opening angle, as a function of p?

   (b) This type of decay is often measured using an "electromagnetic calorimeter", a device which absorbs nearly all of the energy in the incident photon, and produces a signal proportional to the energy. The process of absorbing the energy of a high energy photon involves a cascade or "shower" of production of e^+e^- pairs and new photons as the particles interact with the absorbing medium.

   The CLEO experiment (at the e^+e^- colliding-beam storage CESR) has a calorimeter constructed from approximately 10,000 CsI(Tl) crystals. The thallium is a small doping to increase scintillation light yield. These crystals are arranged in a cylindrical pattern around the beamline, with circular plugs at the ends. The inner radius of the cylinder is approximately 1 m. The crystal dimension at the inner face is approximately 6 cm (OK, I'm guessing a bit here – I don't have the specifications handy).

The transverse size of an electromagnetic shower in a material is characterized by the "Molière radius", the radius of a cylinder which contains 90% of the shower energy, on the average (see RPP 2004 Eqn. 27.31, http://pdg.lbl.gov/2004/reviews/passagerpp.pdf for the official definition). For CsI(Tl), this radius is $R_M = 3.8\,\mathrm{cm}$, so the transverse shower containment scale is approximately the same as the CLEO crystal size.

The π^0 pattern recognition problem changes in character as the showers of the two decay photons start to merge in the detector. Roughly speaking, this happens in CLEO when the two photons are about a crystal apart. At 90°, for what energy π^0 does this merging start to occur?

10. A simple exercise on flux: Suppose that we have a perfectly elastic rubber ball which bounces back-and-forth between two walls of a box. Let the ball's speed be V_b (a constant). Now suppose we shoot a projectile, with speed V_p into the box, colinear with the ball, so that it collides with the ball. Assume that we shoot the projectile at a random time, as far as the ball's motion is concerned. What is the probability that the projectile will collide with the ball when their relative velocity is $V_p + V_b$?

    Now consider a bubble chamber, filled with liquid deuterium. We shoot a beam of π^- mesons into the bubble chamber. Assume that we are doing a "high energy" experiment, so that the deBroglie wavelength of the π^- is small compared to the typical $p - n$ separation in the deuteron. Thus, sometimes the π^- will hit the neutron in the deuteron, and knock it out, without the proton being much involved. In such a reaction, we refer to the proton as a "spectator". A spectator proton, after such an interaction keeps moving however it was moving (according to the deuteron wave function) before the interaction, only now it is a free particle, and typically leaves a short visible track in the bubble chamber. Let θ be the angle between the spectator proton track and the π^- beam. Will $\cos\theta > 0$ or $\cos\theta < 0$ most of the time? (This is too easy ... don't get it wrong! Many people have trouble because they don't think enough about what the most important physics is.)

11. Consider the luminosity for a colliding beam experiment. Assume that the the bunches of the two beams have Gaussian density distributions, without correlations in each of the three spatial directions, letting z be the beam direction:

$$\rho_i(\mathbf{x}, t) = \frac{N_i}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{(z \pm vt)^2}{\sigma_z^2}\right)\right], \qquad (28)$$

where N_i is the total number of paricles in a bunch in beam i. Assume further that the bunches of the two beams have the same density distributions.

(a) Determine the luminosity of the accelerator, averaged over the bunch structure, in terms of the bunch size parameters, N_i, and the bunch crossing frequency. You may assume that the collisions are colinear, and that the variation in bunch size with orbit position may be neglected.

(b) Compute the luminosity of an e^+e^- collider with the following characteristics: Symmetric in the two beams $E_+ = E_- = 5\,\text{GeV}$; $\sigma_x = 500\,\mu\text{m}$; $\sigma_y = 11\,\mu\text{m}$; $N_b = 17 \times 10^{10}$; ring circumference $= 768\,\text{m}$, and 27 bunches in each beam. Determine the integrated luminosity for an experiment which runs at this luminosity for one year, with an efficiency of 1/3. These are roughly the parameters of the CESR storage ring.

12. Now consider the luminosity of a fixed-target experiment.

What is the integrated luminosity of a hydrogen bubble chamber experiment with the following conditions: Fiducial region of chamber has length 1 m; 20 GeV pion beam with an average of 10 particles per pulse, and a pulse repetition rate of 1 Hz; Total effective running time of experiment of 6 months. Express your answer in sensible particle physics units.

13. Let us pursue the topic of beam optics a bit further, this time with a discussion of focussing elements. Quadrupole magnets are typically used for focussing of charged particle beams. Consider such a magnet, of length d and field gradient g. You may make the thin-lens approximation. Of course, a quadrupole which is "focussing" in one transverse dimension is "defocussing" in the other dimension.

    (a) What is the "focal length" F of this magnet, for a positron of momentum p? You may assume small deflections. To get an idea of practical quantities, what is the focal length in meters for $d = 50\,\mathrm{cm}$, $g = 1\,\mathrm{T/m}$, and $p = 10\,\mathrm{GeV}$?

    (b) What is the transfer matrix, in one dimension, for a quadrupole magnet, in the thin-lens approximation? Differently from the previous problem, parametrize your matrix with the momentum p, and express it as a matrix operator in the two-dimensional (x, x') phase space, where $x' \equiv dx/ds$, and s is the coordinate along the magnet axis. You may assume small deflections, and trajectories nearly parallel with the magnet axis. Express your matrix in terms of the focal length F.

    (c) To build a beam transport lattice, we must also include the field-free "drift" regions. What is the transfer matrix for a drift region of length L?

    (d) Now let's put these elements together into a "module" that will have desirable properties in both transverse dimensions, though it will be sufficient here to continue to work in a single transverse dimension. Thus, construct a module built from an initial drift region of length $L/2$, a focussing thin quadrupole of focal length F, another drift region of length L, a defocussing thin quadrupole of focal length $-F$, and finally a drift region of length $L/2$. Note that we could imagine putting such modules one after another to build a lengthy beam transport. What is the transfer matrix of this "FODO" cell?

    (e) Suppose we build a FODO cell with, at a specified momentum, $L/F = \sqrt{2}$. This is called a "quarter-wave" cell. To see why, determine the transfer matrix for two of these cells placed in succession. Convince yourself that the result holds for both transverse dimensions. Thus, we have arrived at optics which, in the linear approximation, and for mono-chromatic beams, can transport a beam for long distances. This is an important result, and provides a foundation for all of our modern high energy accelerators.