Physics 231a
Problem Set Number 3
Due Wednesday, October 20, 2004

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

14. Standard Model (review?): The QCD Lagrangian. We may write down the free quark Lagrangian (for a given flavor), equivalent to the Dirac equation for the quark field:

$$\mathcal{L}_0 = \bar{q}_j(x)[i\gamma^\mu\partial_\mu - m]q_j(x)$$

where the sum over colors, $j = 1, 2, 3$, is implied. We typically leave off this subscript for simplicity, and let q be a three-vector in color space. We would like to find a Lagrangian with this structure (*i.e.*, which gives us the Dirac equation) which is invariant under local gauge transformations $q(x) \to U(x)q(x) = e^{i\alpha_a(x)T_a}q(x)$. [Implied sum over $a = 1, 2, ..., 8$, where T_a are the generators of $SU(3)$]. We proceed by analogy with QED. (Note that $\mathcal{L}_0$ does not have this invariance.) One approach is to introduce a "covariant derivative", which depends on the color gauge fields according to:

$$D_\mu = \partial_\mu + igT_aG^a_\mu$$

(a) Show that the desired invariance of $\mathcal{L}$ is obtained if we substitute $\partial_\mu \to D_\mu$ in $\mathcal{L}_0$:

$$\mathcal{L} = \bar{q}(x)[i\gamma^\mu D_\mu - m]q(x),$$

and if the covariant derivative has the property:

$$\begin{aligned} &\text{If} \quad q(x) \to e^{i\alpha_a(x)T_a}q(x), \\ &\text{then} \quad D_\mu q(x) \to e^{i\alpha_a(x)T_a}D_\mu q(x). \end{aligned}$$

(b) By considering infinitesimal gauge transformations, show that the gauge transformation law for the gauge fields must be:

$$G_\mu^a \to G_\mu^a - \frac{1}{g}\partial_\mu \alpha_a - f_{abc}\alpha_b G_\mu^c,$$

in order to have gauge invariance of the Lagrangian. The f_{abc} are the structure constants of $SU(3)$, defined by $[T_a, T_b] = if_{abc}T_c$. Try to "derive" the above, rather than simply "verifying" that it works – *i.e.*, show that there really isn't any freedom involved.

(c) Plugging in the covariant derivative $D_\mu = \partial_\mu + igT_a G_\mu^a$, we arrive at the Lagrangian:

$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - m)q - g\bar{q}\gamma^\mu T_a q G_\mu^a.$$

The first term corresponds to the quark kinetic energy, and the second term to the interaction between quarks and gluons (the gauge particles, analogues of the photon). But, as in QED, we should add a term to take account of the kinetic energy of the gluons. In QED, we take account of the photon kinetic energy by adding a scalar constructed from the gauge invariant field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In QED, we could equivalently have defined:

$$[D_\mu, D_\nu] = -ieF_{\mu\nu}.$$

By analogy, for QCD define:

$$[D_\mu, D_\nu] = igT_a G_{\mu\nu}^a.$$

[Do you know where the sign difference comes from?] Show that this yields:

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - gf_{abc}G_\mu^b G_\nu^c$$

(d) Finally, show that the complete QCD Lagrangian:

$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - m)q - g(\bar{q}\gamma^\mu T_a q)G_\mu^a - \frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu}$$

is gauge invariant under local gauge transformations. You have already demonstrated this for the first two terms, so you need only worry about the gluon field "kinetic energy" term (which includes the gluon-gluon interactions) here. Rather than attempt to grind it out, I suggest you first consider how the covariant derivative transforms under a gauge transformation, and what this implies for the transformation of the gluon field strength tensor. Then notice that the gluon "kinetic energy" term is a trace. [Note that we haven't actually proven here that we got the right "kinetic energy" term for the gluons.]

15. The π^0 mass is 134.98 MeV, while the $\pi^\pm$ mass is 139.57 MeV. We suspect that the mass difference may be electromagnetic in origin. Suppose that the mass difference is due to the different Coulomb potential energies. Make an approximate estimate in non-relativistic quantum mechanics for the size of the pion assuming this is the explanation. Comment on your calculation and result.

16. The K^0 meson is built, in the valence quark model, from a down quark and a strange anti-quark. Give the wavefunction of the K^0 meson in terms of the flavor, spin, and color of the constituent quarks. [A note on convention: The K^0 meson has strangeness $S = +1$, but is made with an $\bar{s}$ quark. Hence, the strange quark has strangeness $S = -1$. This is a bit of historical accident, but has now become the basis for the following convention: The sign of the quark flavor is the same as the sign of its electric charge. Hence, the charm quark has charm $C = +1$, the bottom quark has bottomness $B = -1$, *etc.* This convention has become relatively widely adopted, but older papers often adopt a convention that the bottom quark has $B = +1$.]

17. The J/ψ has branching ratios to all-pion final states (from RPP, 2004):

Final state	Branching fraction
$\pi^+\pi^-$	$(1.47 \pm 0.23) \times 10^{-4}$
$\pi^+\pi^-\pi^0$	$(1.50 \pm 0.20) \times 10^{-2}$
$2(\pi^+\pi^-)$	$(4.0 \pm 1.0) \times 10^{-3}$
$2(\pi^+\pi^-)\pi^0$	$(3.37 \pm 0.26) \times 10^{-2}$
$3(\pi^+\pi^-)$	$(4.0 \pm 2.0) \times 10^{-3}$
$3(\pi^+\pi^-)\pi^0$	$(2.9 \pm 0.6) \times 10^{-2}$

Give a qualitative explanation for the most obvious pattern in these numbers.

18. An approximate SU(6) flavor-spin symmetry (of the strong interaction) is obtained by considering the basis states consisting of the u, d, s quarks, each with its two spin polarization states. [Another SU(6) symmetry, though badly broken by the masses, corresponds to transofrmations among the 6 known quarks, u, d, s, c, b, t.] Study the discussion of Young's diagrams at http://pdg.lbl.gov/2004/reviews/youngrpp.pdf and use them to find the dimensions (*i.e.*, numbers of particle states) of the irreducible representations of $SU(6)$ corresponding to the baryons. Noting that a Young diagram consisting of a horizontal row corresponds to a symmetric representation (S), a vertical column to an antisymmetric (A), and others to mixed symmetries (M), subscript your results with the appropriate symmetry labels. (If you haven't studied them before, and you want a more fundamental understanding of Young Tableaux, see Hamermesh or Wu-Ki Tung).

19. We mentioned the predicted "GZK" cut-off in the cosmic ray energy spectrum in class.

    (a) Consider the collision of a high energy cosmic ray proton with a three-degree microwave background photon. What energy proton is required to produce a Δ^+ resonance?

    (b) The cross section for $a + b \to R$ scattering near a resonance R with angular momentum J is given approximately by the "Breit-Wigner" formula:

    $$\sigma(E) = \frac{n_J}{n_a n_b} \frac{\pi}{k^2} \frac{B_i B_f \Gamma_T^2}{(E - M)^2 + \Gamma_T^2/4}, \tag{29}$$

    where n_j is the spin statistical weight (usually given by $2s_j + 1$, but is just 2 for photons), k is the momentum of a in the CM frame, B_i is the branching fraction for the initial state (i.e., the branching fraction for $R \to a + b$), B_f is the branching fraction for the final state (equal to 1 if all decays of R are included), E is the total energy in the CM frame, M is the resonance mass (loosely), and Γ_T is its total width. [If you have never derived this formula in time-dependent perturbation theory, you should review it!]

    Determine the cross section for $p\gamma \to \Delta^+$ at the peak of the resonance.

(c) Finally, determine (very approximately, in principle an integral should be done over the black body spectrum, but I am not asking for that here) the mean free path of a cosmic ray photon with energy from part (a), in the microwave background. Compare with some relevant astrophysical scale.