Physics 231a
Problem Set Number 5
Due Wednesday, November 3, 2004

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

25. Standard Model Review(?): As a simple prototype of the problem of massive gauge bosons in the weak interaction, let us suppose that we want to give the photon in QED a mass. We start with the QED Lagrangian for an electron in an electromagnetic field:

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad (30)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Under a $U(1)$ gauge transformation, $\psi \to \psi' = e^{i\theta(x)}\psi$, the photon field transforms (as you worked out in problem 7) according to:

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta. \qquad (31)$$

(a) Show that the addition of a mass term, $\frac{1}{2}M^2 A_\mu A^\mu$, for the photon would destroy gauge invariance. [So, don't add it!]

(b) Now add a complex scalar field, $\phi = \phi_1 + i\phi_2$, to your theory, with Lagrangian:

$$\mathcal{L}_\phi = \frac{1}{2}\partial^\mu\phi^\dagger\partial_\mu\phi + \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{1}{4}\lambda^2\left(\phi^\dagger\phi\right)^2. \qquad (32)$$

This Lagrangian has global $U(1)$ gauge invariance under $\phi \to e^{i\theta}\phi$, but not local gauge invariance. Fix this by adding an interaction with the photon field.

(c) Note that the "mass term" in $\mathcal{L}_\phi$ has been inserted with a reversed sign, that is $\mu^2 < 0$ if this is to be interpreted as corresponding to a real mass for our field. But we want to consider here the possibility of $\mu^2 > 0$. Thinking along the lines of the Lagrangian as being the kinetic minus the potential energy, define the scalar potential:

$$V(\phi) \equiv -\frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{4}\lambda^2\left(\phi^\dagger\phi\right)^2. \tag{33}$$

At what field value(s) is this potential minimized?

(d) We pick a ground state to do our perturbation theory about. Define new fields $\eta = \eta_1 + i\eta_2$ according to

$$\phi = \eta_1 - v + i\eta_2, \tag{34}$$

such that v is real and positive, and the point $\eta = 0$ corresponds to a minimum of the potential. Rewrite the Lagrangian for the scalar field in terms of η, calling it $\mathcal{L}_\eta$. Be sure to include the photon terms (but you should find that you don't need to be concerned in this discussion about the electron terms).

(e) If you have done things all right so far, your Lagrangian will have a massless scalar field (and a massive one), as well as a strange "off-diagonal" coupling between two fields. The massless field is called a "Goldstone Boson". Get rid of both of these troubling terms by making a suitable gauge transformation.

(f) You should end up with a mass term for the photon, what is its mass? Note that the underlying theory still has local $U(1)$ gauge invariance; we have "spontaneously broken" it by choosing a particular ground state ("vacuum"). Note that the degree of freedom corresponding to the Goldstone field is still there – it has just been transformed into a longitudinal degree of freedom for the photon.

26. Quark model: By considering color SU(3) representations, determine whether you would expect to observe $qq\bar{q}$ states in nature, according to the quark model of hadrons.

27. In class, I mentioned that the reaction $\pi^- d \to nn$ was used to determine the parity of the charged pion.

An important assumption in the argument is that the parities of the proton and neutron are the same. Note that this is really a convention – there is no way to compare the relative parities of the proton and neutron, as long as electric charge is conserved. Charge conservation implies a "superselection rule", where the parities of particles could be modified according to

$$\eta_a \to \eta'_a = \eta_a e^{i\theta Q_a},$$

where a is the particle label, η_a is its parity, and Q_a is its electric charge. By charge conservation, such a transformation will not alter any observables.

The reaction $\pi^- d \to nn$ is observed to occur for stopping pions in deuterium, at a faster rate than would be expected for a weak interaction. Assume that the reaction occurs from an S-wave $\pi^- d$ state (mesonic atom). This assumption must be justified, of course, and has been, both theoretically and experimentally. Hence, determine the parity of the charged pion. What would your result be if the proton and neutron had opposite parities?

Note that the π^0 parity is measured independently, and by the same superselection rule, need not be the same as that of the charged pion. One way that the π^0 parity is determined is by an analysis of the angular distribution in double-Dalitz decay, $\pi^0 \to e^+e^-e^+e^-$.

28. Neutral K mesons: In class we noted that the K^0 meson had two lifetimes, and suggested that this could occur if the mass eigenstates were not eigenstates under the weak decay Hamiltonian. For a number of reasons, this is a very important system, as is the somewhat similar B^0 meson system, and in this problem we get a start on understanding the phenomenology. In our discussion of the quark model, we found the states in our pseudoscalar octet:

$$|K^0\rangle = |d\bar{s}\rangle,\ |\bar{K^0}\rangle = |\bar{d}s\rangle$$

If we define a strangeness operator, S, then these states are eigenstates, with:

$$S|K^0\rangle = |K^0\rangle$$
$$S|\bar{K^0}\rangle = -|\bar{K^0}\rangle.$$

S can be written as the 2×2 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in the $|K^0\rangle, |\bar{K^0}\rangle$ basis, but a convenient "basis-independent" form is:

$$S = |K^0\rangle\langle K^0| - |\bar{K^0}\rangle\langle\bar{K^0}|$$

These are not eigenstates of C, the charge conjugation operator. It is conventional to pick the antiparticle phases such that:

$$C|K^0\rangle = -|\bar{K}^0\rangle, \qquad C|\bar{K}^0\rangle = -|K^0\rangle$$

Thus, if we multiply C by the parity operator, P:

$$CP|K^0\rangle = |\bar{K}^0\rangle$$

$$CP|\bar{K}^0\rangle = |K^0\rangle$$

(a) Express the CP operator for neutral kaons in a similar "basis-independent" form as we did for S. Find the normalized eigenstates of CP and the eigenvalues. Express your eigenstates in terms of linear combination of $K^0, \bar{K}^0$, calling the state with the positive eigenvalue K^0_S, and the other one K^0_L. Are your eigenstates orthogonal?

(b) A neutral kaon can only decay via the weak interaction. Why?

Let us assume that CP is conserved in the weak interaction (experimentally this is true to a good approximation. In fact, the only observed violation is in the neutral kaon system!). Referring to your particle data tables, you should find that the K^0_S typically decays to two pions while the K^0_L typically does not, but often decays into 3 pions. Furthermore, the lifetimes of K^0_S and K^0_L are quite different, and their masses are slightly different.

(c) Why did I specify in part a) that the state with positive CP eigenvalue was to be called K^0_S?

According to quantum mechanics, a particle is described by a probability amplitude with a phase which varies in time with a frequency given by:

$$\omega = E = \sqrt{m^2 + p^2}$$

(It is only the kinetic energy portion of this which is of importance in most ordinary QM. In this case, however, we must keep the rest mass contribution. I am, of course, assuming that we are dealing with a free particle.) Thus, at a given momentum, p, the frequencies of a K_S^0 and a K_L^0 wave are slightly different due to the mass difference:

$$\omega_S = \sqrt{p^2 + m_S^2}$$

$$\omega_L = \sqrt{p^2 + m_L^2}$$

In addition to this phase, the time dependence of the probability amplitude for a neutral kaon must take into account the fact that the particle decays. Thus, if at $t = 0$ we start with a K_S^0:

$$\psi(0) = |K_S^0\rangle$$

then, at time t we have:

$$\psi(t) = \exp[-i\omega_S t - t/2\tau_S]|K_S^0\rangle,$$

where τ_s is the mean lifetime of K_S^0. Likewise, if $\psi(0) = |K_L^0\rangle$, then

$$\psi(t) = \exp[-i\omega_L t - t/2\tau_L]|K_L^0\rangle.$$

(d) Suppose at $t = 0$, we have a pure $\bar{K}^0$ state:

$$\psi(0) = |\bar{K}^0\rangle.$$

(Neutral kaons are generally produced via the strong interaction, which conserves strangeness, so this is an experimentally reasonable possibility. If you find this comment cryptic – think about it.) Find the probability $P_{K^0}(t)$ that a K^0 meson will be observed at some later time t. ($P_{K^0}(0) = 0$, of course).

Considering the known parameters of the neutral kaons, what is the approximate maximum value $P_{K^0}(t)$ attains, and at approximately what time?

Note the remarkable nature of this – you start with a strange quark and later may observe instead an anti-strange quark. This is the fault of the weak interaction, which we will study in more detail later.

29. Just as there are constants which descibe the "coupling" to angular momentum states when combining angular momenta (or isospin) ("Clebsch-Gordan coefficients"), there are analogous constants for SU(3). We could go through the calculations necessary to derive these constants, as we do in quantum mechanics courses for SU(2), but let us instead just look them up and use them to determine the relative rates for two processes related by SU(3).

    (a) Determine, up to phase space factors, the relative rate for an $8 \rightarrow 10 \otimes 8$ process $\Xi^0 \rightarrow \Sigma^+ K^-$ relative to $\Lambda^0 \rightarrow \Xi^- K^+$. You may use the discussion of SU(3) Isoscalar Factors at: http://pdg.lbl.gov/2004/reviews/su3rpp.pdf.

    (b) What do you expect to be the relative rate for

$$\frac{J/\psi \rightarrow \Delta^+ \bar{p}}{J/\psi \rightarrow p\bar{p}}?$$