

Physics 231a
 Problem Set Number 7
 Due Wednesday, November 17, 2004

Note: Some problems may be “review” for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write “been there, done that”, or suitable equivalent, for that problem, and I’ll give you credit.

35. Standard Model Review(?): We have reviewed the separate $U(1)$, $SU(3)$, and $SU(2)$ gauge group ingredients in the standard model. However, our $SU(2)$ model for the weak interactions is not yet complete, for several reasons:

- The mass of the neutral weak boson, the Z^0 , is observed to be different from the charged weak bosons, the W^+ and W^- . If these form a weak isospin triplet, the masses should be equal.
- The charged weak boson is observed to couple to fermions with a “left-handed” coupling. That is, the $f\bar{f}W^\pm$ interaction term in the Lagrangian is of the form:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}}\bar{\psi}\frac{1}{2}(1 - \gamma^5)\gamma^\mu(T^+W_\mu^+ + T^-W_\mu^-)\psi, \quad (44)$$

where T^+ is the weak isospin raising operator (for example, $T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in a two-dimensional representation), T^- is the weak isospin lowering operator, and

$$W^\pm \equiv \frac{1}{\sqrt{2}}(W_1 \mp iW_2). \quad (45)$$

- The neutral weak boson is observed to have both left- and right-handed couplings with the fermions.

There are some other issues as well, but let us concentrate on one of the modifications required to accommodate these for now.

The fact that $m_{Z^0} \neq m_{W^\pm}$ (we’ll simplify notation to simply m_Z and m_W) suggests that the physical Z^0 cannot be simply identified as the

weak $SU(2)$ gauge particle W_3 . The standard model answer is to say that the physical Z^0 (or simply Z) and photon (A) states are mixtures of neutral gauge bosons. Thus, in the standard model we start with gauge groups “ $SU(2)_L$ ” and “ $U(1)_Y$ ”, where the “ L ” subscript stands for left-handed, and the “ Y ” stands for (weak) hypercharge. The gauge bosons of $SU(2)_L$ are the W_1, W_2, W_3 , all with only left-handed coupling to fermions. The $U(1)_Y$ gauge boson is denoted B . The Z and A fields are the mixtures:

$$A = B \cos \theta_W + W_3 \sin \theta_W \quad (46)$$

$$Z = -B \sin \theta_W + W_3 \cos \theta_W, \quad (47)$$

where θ_W is the “weak mixing angle” (or sometimes “Weinberg angle”). The Higgs mechanism that you worked out last week gives a mass term in the Lagrangian for the W fields. We assume there is no mass term for the B field.

- (a) According to this mixing hypothesis, determine the mass of the Z in terms of the mass of the charged W and the mixing angle. You may assume that the photon is massless, of course.
- (b) Determine the coupling (interaction Lagrangian) for the $Zff\bar{f}$ vertex in this model, and relate e , g , and θ_W .

Note that this model may be tested: We can compare the mixing angle as measured by the masses with the mixing angle as measured by the couplings – we should get the same result for the angle.

36. Consider an e^+e^- colliding beam storage ring. The term “luminosity lifetime” may be used to refer to the time that the luminosity takes to decrease from its peak (\mathcal{L}_0) to a value $1/e$ of the peak (people use this term in other ways as well). There are various contributions to the decrease in luminosity with time. Let us consider one such contribution – the loss of particles due to elastic e^+e^- (Bhabha) scattering. Sometimes such a scattering will be at a large enough angle that the particle will be lost from the beam, and the luminosity will be reduced as a result. Let σ be the cross section to lose a particle by this mechanism.
 - (a) Suppose that the machine is operated in such a way that the spot sizes at the interaction point are held constant. Assume also a

“symmetric” machine, *i.e.*, with identical parameters (with the obvious exceptions) for the two beams. Let N_0 be the number of particles in one beam at maximum luminosity (time 0). Obtain a formula for the time dependence of the luminosity.

The differential cross section for $e^+e^- \rightarrow e^+e^-$, in lowest order perturbation theory, is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left[\frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} - 2 \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \frac{1}{2}(1 + \cos^2 \theta) \right],$$

where θ is the scattering angle in the center of mass. You should be comfortable that you know how to calculate this result.

- (b) To actually calculate the luminosity lifetime due to Bhabha scattering, we need to determine the minimum scattering angle which leads to loss of a beam particle. To do this, we consider the typical angles of the trajectories of the beam particles, given by the emittance (ϵ) and the value of the β function at the point of interest. Suppose that a particle is lost if it scatters through an angle equal to or greater than n standard deviations of the distribution of normal (no scattering effects) trajectory angles. For simplicity, suppose also that this angle is the same in both horizontal and vertical planes (storage rings are often designed such that this is not a bad approximation at the IP). Determine this minimum angle in terms of ϵ , β , and n , and give the cross section to lose a particle in terms of this minimum angle.
- (c) Now let us apply all this to a slightly unrealistic, but hopefully illuminative, example: Calculate the Bhabha luminosity lifetime for a machine with $\mathcal{L}_0 = 10^{34} \text{ cm}^{-2}\text{s}^{-1}$, $N_0 = 4 \times 10^{13}$, $E_{\text{cm}} = 10 \text{ GeV}$, $\beta_x^* = 0.5 \text{ m}$, and $\epsilon_x = 50 \text{ nm-rad}$. You may assume that there is zero dispersion at the IP.

37. Let us discuss some of the issues relevant to the proof you gave of Yang’s theorem. In particular, in this problem, we consider the parity again, and try to establish the connection between how I expected you to do the homework, and an equivalent argument based on the identical boson symmetry.

- (a) In general, the parity of a system of two particles, when their state is an eigenstate of the parity operator, may be expressed in the form:

$$P = \eta_{\text{intrinsic}} \eta_{\text{spatial}},$$

where $P^2 = 1$, $\eta_{\text{intrinsic}}$ refers to any intrinsic parity due to the reflections of the positions of the particles, and η_{spatial} refers to the effect of parity on the spatial part of the wave function. Given a system of two identical particles, and considering the action of parity on $Y_{L0}(\theta, \phi)$, determine the parity of the system for given orbital angular momentum L .

- (b) For a two-photon system, such as we considered in our discussion of Yang's theorem, use the boson symmetry to once again determine the parity of the states:

$$|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle], \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

Also, give the allowed possibilities for the orbital angular momentum for these states, at least at the level of our current discussion. You may now wish to go back to your original derivation of Yang's theorem, and determine where you implicitly made use of the boson symmetry.

- (c) Continuing our discussion of Yang's theorem, there may be some concern about the total spin angular momentum of the two photon states, and whether the appropriate values are possible to give the right overall angular momentum when combined with even or odd orbital angular momenta. Using a table of Clebsch-Gordan coefficients or otherwise, let us try to alleviate this concern. Thus, decompose our four 2-photon helicity states (with J_z values indicated by $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle], \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$, where the photons are travelling along the $+$ and $-z$ axis) into states of total spin angular momenta and spin projection along the z -axis: $|S, S_z\rangle$. Hence, show that a $J^P = 0^+$ particle may decay into two photons with relative orbital angular momentum $L = 2$ or 0 , and a $J^P = 0^-$ particle may decay into two photons with relative angular momentum $L = 1$.

38. **Dalitz plots:** An important analysis tool in particle physics has been the Dalitz plot (after R.H. Dalitz, *e.g.*, *Rev. Mod. Phys.* **31** 823

(1959)). Quite often we are led to consider, experimentally and phenomenologically, a three particle final state, or a three particle decay of some resonance. Let us see what a Dalitz plot is, and get an idea of why it is a useful tool. We consider the process $X \rightarrow 1 + 2 + 3$, where initial state X goes to three particles, 1,2, and 3. Recall (*e.g.*, chapter 4 of Halzen & Martin) that the differential decay rate (if X is a particle) may be written:

$$d\Gamma = \frac{1}{2m_X} |\mathcal{M}|^2 dLIPS,$$

in the X rest frame, with

$$dLIPS = (2\pi)^4 \delta^{(4)}(p_X - p_1 - p_2 - p_3) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}.$$

- (a) We suppose we do an experiment where we measure events in the above decay. Show that the density of points on a scatterplot of E_1 vs. E_2 is proportional to

$$|\mathcal{M}(E_1, E_2)|^2 dE_1 dE_2$$

In particular, notice that if the invariant matrix element is a constant, then the graph has a uniform density of points within the kinematic boundaries.

- (b) If m_{ij} is the invariant mass of particles i and j (*i.e.*, $m_{ij}^2 = (p_i + p_j)^2$), show that an equivalent approach is to make a graph of m_{13}^2 vs. m_{23}^2 (*i.e.*, a uniform density in $dE_1 dE_2 \Leftrightarrow$ uniform density in $dm_{13}^2 dm_{23}^2$).
- (c) Make a careful picture showing the kinematic boundary for the Dalitz plot with axes m_{13}^2 and m_{23}^2 for the decay $J/\psi \rightarrow \gamma\gamma\gamma$. Study the Dalitz plot below (from Phys. Rev. Lett. **11** (1980) 712) and discuss the structure. You may wish to understand the boundary drawn – note that the bottom boundary line depends on the

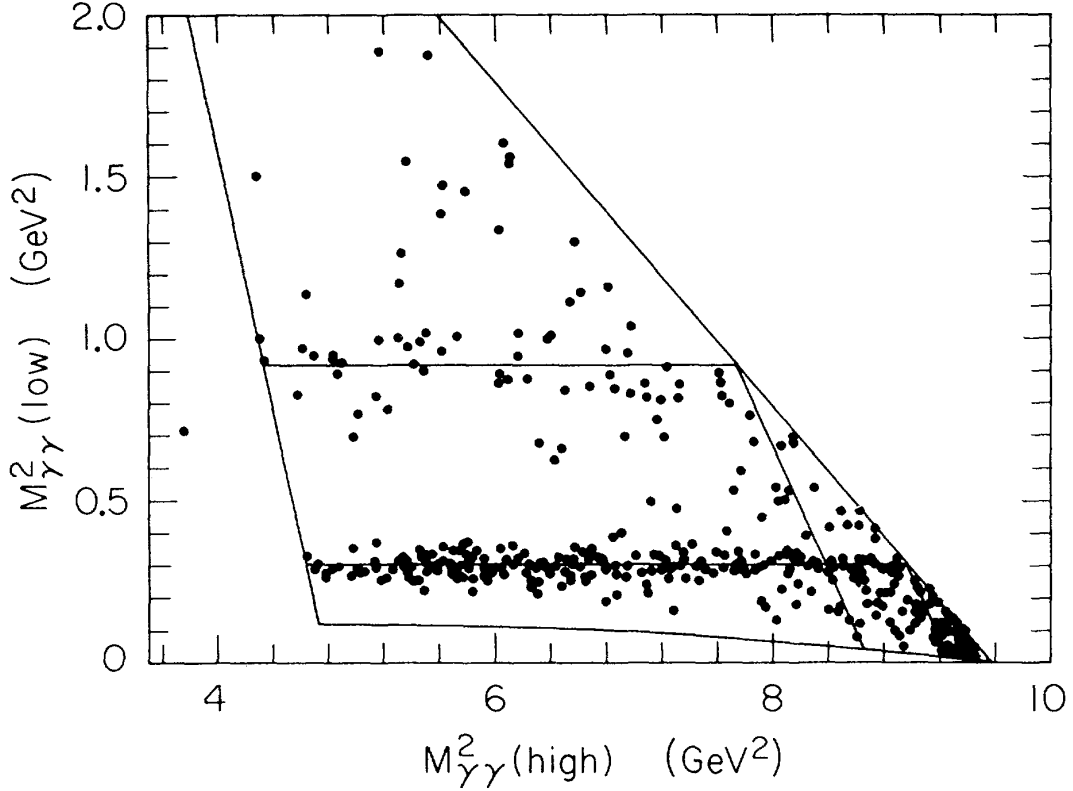


FIG. 2. Dalitz plot of $J/\psi \rightarrow 3\gamma$.

39. Resonant scattering: Let us consider the elastic scattering of a plane wave (“particle A”) on a fixed scattering center (“particle B”) located at the origin. Denote the incident plane wave by:

$$\psi_0(\vec{x}, t) = N e^{i\vec{p}_0 \cdot \vec{x} - i\omega t}$$

The scattered wave will be the result of some diffractive process on the scattering center. What is the wave function of the scattered wave? Well, it is supposed to be an elastic scatter, so it must oscillate with frequency ω . Let us assume that the process is linear, so that the strength of the scattered wave is proportional to the strength of the incident wave, *i.e.*, to N . We will be interested here in the form of the scattered wave far from the scattering center, so we expect, up to some

angular variation, that the scattered wave will look like an outgoing spherical wave. Finally, by the assumed symmetry of our problem (*e.g.*, we could be describing the scattering of spinless particles, or of particles with spin but where we average over all possible spin orientations), the angular dependence of the scattered wave can be written as a function only of the angle between the scattered direction and the incident wave, which we call θ .

Thus, let us write the scattered wave, far from the scattering center:

$$\psi_s(\vec{x}, t) \cong N f(\theta; p) \frac{e^{ipr - i\omega t}}{r}, \quad r \equiv |\vec{x}|.$$

We refer to the function $f(\theta)$ as the “scattering amplitude”.

- (a) The differential cross section $\frac{d\sigma}{d\Omega}$ is defined as the probability per unit of incident probability flux (*i.e.*, incident probability per unit area) for the particle to emerge within element of solid angle $d\Omega$ around angle (θ, φ) . By elementary arguments, determine $\frac{d\sigma}{d\Omega}$ in terms of the quantities we have defined already. A cross section, of course, has units of area.

Let us suppose that any spatial extent of our scattering center is small compared to the wavelength of the incident plane wave. In this situation, any “excitation” at the scattering center samples uniformly the same phase of the wave, and hence there will be no angle-dependent interference effects; *i.e.*, $f(\theta; p) = f(p)$, independent of θ . Far from the origin, we have the total amplitude:

$$\psi(\vec{x}, t) \cong N e^{i\vec{p}_0 \cdot \vec{x} - i\omega t} + N f(p) \frac{e^{ipr - i\omega t}}{r}$$

- (b) We wish to investigate $f(p)$ further. To do this, it is convenient to reformulate our problem in a different, but equivalent, form. Since $f(\theta; p)$ does not depend on θ , the scattered wave is the same if we take our incident wave to be the average over angles of plane waves in all directions:

$$\psi'_0(\vec{x}, t) = \frac{1}{4\pi} \int_{(4\pi)} d\Omega N e^{i\vec{p}_0 \cdot \vec{x} - i\omega t}$$

Show that ψ'_0 may be written as the superposition of incoming and outgoing spherical waves, and hence that the total wave is

$$\psi'(\vec{x}, t) \cong \frac{N}{2ipr} \{ [1 + 2ipf(p)]e^{ipr} - e^{-ipr} \} e^{-i\omega t}$$

far from the origin.

Now we can make an important observation: The relative magnitude of the ingoing and outgoing spherical waves describes whether our type “A” particles are being preferentially produced or destroyed at the scattering center. In the case of elastic scattering, the number of particles of type A is conserved; hence

$$|1 + 2ipf(p)| = 1$$

It is thus conventional to write (for elastic scattering):

$$f(p) = \frac{1}{2ip}(e^{2i\delta} - 1)$$

where $\delta = \delta(p)$ is called the (s-wave) phase shift.

- (c) What is the maximum value that a spherically symmetric (s-wave) elastic scattering cross section, $\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega} d\Omega$, can have? Interpret your answer “geometrically” in terms of the deBroglie wavelength of the incident particle.
- (d) For a particle of definite mass, $p = p(\omega)$, and hence we can write $\delta = \delta(\omega)$, $f = f(\omega)$. Show that we may write:

$$f(\omega) = \frac{1}{p} \frac{1}{\cot[\delta(\omega)] - i}$$

Note that at places where $\delta(\omega) = (n + \frac{1}{2})\pi$, the scattering amplitude, and hence the cross section, achieves a maximum. The scattering is called “resonant” at any energy (ω_0 , say) such that this condition occurs [$\delta(\omega_0) = (n_0 + \frac{1}{2})\pi$]. Finally, show that, for energies near the resonance, the scattering amplitude is:

$$f(\omega) \cong -\frac{1}{p} \frac{\Gamma/2}{(\omega - \omega_0) + i\Gamma/2}, \quad \omega \approx \omega_0$$

where $\frac{d \cot \delta(\omega)}{d\omega} \Big|_{\omega=\omega_0} \equiv -\frac{2}{\Gamma}$. Express the cross section $\sigma = \sigma(\omega)$ as a function (Breit-Wigner!) of ω , for $\omega \approx \omega_0$.