Physics 231a
Problem Set Number 8
Due Wednesday, November 24, 2004

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

40. Standard Model Review(?): Last week you considered the mass matrix and Z coupling for the neutral gauge bosons in the electroweak theory. Let us discuss a little more completely the couplings of the electroweak gauge bosons to fermions.

    Again, we'll work in the standard model where the physical Z and photon (A) states are mixtures of neutral gauge bosons. We start with gauge groups "$SU(2)_L$" and "$U(1)_Y$". The gauge bosons of $SU(2)_L$ are the W_1, W_2, W_3, all with only left-handed coupling to fermions. The $U(1)_Y$ gauge boson is denoted B. The Z and A fields are the mixtures:

$$\begin{aligned} A &= B\cos\theta_W + W_3\sin\theta_W & (48)\\ Z &= -B\sin\theta_W + W_3\cos\theta_W, & (49) \end{aligned}$$

    where θ_W is the "weak mixing angle".

    The Lagrangian contains interaction terms with fermions of the form:

$$\mathcal{L}_{\text{int}} = -g\bar{f}_L\gamma^\mu\frac{1}{2}\tau\cdot W_\mu f_L - g'\bar{f}\frac{1}{2}YB_\mu\psi, \qquad (50)$$

    where $f_L \equiv \frac{1}{2}(1-\gamma^5)\psi$, $\tau\cdot W_\mu \equiv \sum_{i=1}^{3}\tau_i W_{\mu i}$, τ are the Pauli matrices acting on weak $SU(2)_L$ fermion doublets, Y is the weak hypercharge operator, and g and g' are the interaction strengths for the $SU(2)_L$ and $U(1)_Y$ components, respectively.

    (a) Let's think a little about the charged ($W^\pm$) interaction first. Show that the charged interaction as written here is consistent with the discussion and notation in the statement of problem 35. Consider the electron and electron neutrino (neglect any neutrino mixing for now). Define the weak isospin left-handed doublet:

$$\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L . \tag{51}$$

Show that the charged current interaction can be written as a "current" interaction:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} J_\mu^\pm W^{\mu\pm}, \tag{52}$$

and give an expression for the current $J_\mu^\pm$ in terms of the χ_L fields. These are referred to as "charge-raising" or "charge-lowering" interactions, or collectively as "charged-current" interactions, because the fermion electric charge is changed.

(b) Now consider the "neutral" $SU(2)_L$ current:

$$J_\mu^3 \equiv \bar{\chi}_L \gamma_\mu \frac{1}{2} \tau_3 \chi_L, \tag{53}$$

and the weak hypercharge current:

$$J_\mu^Y \equiv \bar{\psi} \gamma_\mu Y \psi. \tag{54}$$

We define the weak hypercharge operator according to

$$Q = T^3 + \frac{Y}{2}, \tag{55}$$

where Q is the electric charge operator. Thus, the electric current may be written:

$$J_\mu^Q = \bar{\psi} \gamma_\mu Q \psi = J_\mu^3 + \frac{1}{2} J_\mu^Y. \tag{56}$$

Starting with the neutral $SU(2)_L \times U(1)_Y$ interaction terms:

$$-igJ^{3\mu} W_\mu^3 - i\frac{g'}{2} J_\mu^Y B_\mu, \tag{57}$$

derive the neutral current interactions for the photon and the Z. In particular, rewrite the J_μ^Z current in terms of currents J_μ^3 and J_μ^Q.

41. Consider e^+e^- annihilation to two spinless particles A and B via the intermediate production of the $\Upsilon(1S)$ resonance. In the CM frame, what is the angular distribution of the A particle, measured with respect to the incoming electron direction. If A and B are identical particles ($A = B$), is this reaction allowed?

42. Time Reversal in Quantum Mechanics:

    We wish to define an operation of time reversal, denoted by T, in quantum mechanics. We demand that T be a "physically acceptable" transformation, *i.e.*, that transformed states are also elements of the Hilbert space of acceptable wave functions, and that it be consistent with the commutation relations between observables. We also demand that T have the appropriate classical correspondence with the classical time reversal operation.

    Consider a system of structureless ("fundamental") particles and let $\vec{X} = (X_1, X_2, X_3)$ and $\vec{P} = (P_1, P_2, P_3)$ be the position and momentum operators (observables) corresponding to one of the particles in the system. The commutation relations are, of course:

$$[P_m, X_n] = -i\delta_{mn},$$
$$[P_m, P_n] = 0,$$
$$[X_m, X_n] = 0.$$

    The time reversal operation $T : t \to t' = -t$, operating on a state vector gives (in Schrödinger picture – you may consider how to make the equivalent statement in the Heisenberg picture):

$$T|\psi(t)\rangle = |\psi'(t')\rangle.$$

    The time reversal of any operator, Q, representing an observable is then:

$$Q' = TQT^{-1}$$

(a) By considering the commutation relations above, and the obvious classical correspondence for these operators, show that

$$TiT^{-1} = -i.$$

Thus, we conclude that T must contain the complex conjugation operator K:

$$KzK^{-1} = z^*,$$

for any complex number z, we require that T on any state yields another state in the Hilbert space. We can argue that (for you to think about) we can write: $T = UK$, where U is a unitary transformation. If we operate twice on a state with T, then we should restore the original state, up to a phase:

$$T^2 = \eta 1,$$

where η is a pure phase factor (modulus = 1).

(b) Prove that $\eta = \pm 1$. Hence, $T^2 = \pm 1$. Which phase applies in any given physical situation depends on the nature of U, and will turn out to have something to do with spin, as we shall examine in the future.

43. Magnetic moment of electron: Add an electromagnetic field as a perturbation to the Dirac equation. Consider the non-relativistic limit. Evaluate g, the electron "gyromagnetic ratio" in the magnetic moment term for the electron, where the magnetic moment is expressed as:

$$\mu = -g\frac{e}{2m}\mathbf{S}.$$

This is an extremely important result in Dirac theory. High precision tests of the standard model involve evaluating corrections to this value for g at higher orders in perturbation theory, including non-QED contributions, and comparing with experiment. One of the current limiting uncertainties in this comparison is the hadronic vacuum polarization uncertainty in the few GeV regime (where the structure is complicated due to resonances and thresholds, and the experiments have large uncertainties).

44. Evaluate the leading order differential cross section $d\sigma/d\Omega$ for $e^-e^- \rightarrow e^-e^-$ scattering at high energy in the center-of-mass frame. What is the total cross section for this process? Suppose we have a detector with an acceptance of $|\cos\theta < 0.9|$ in the center-of-mass. What luminosity collider is required to obtain an observed collision rate of 10 Hz, at $E_{cm} = 10\,\mathrm{GeV}$?