Physics 231a
Problem Set Number 9
Due Wednesday, December 1, 2004

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

45. Standard Model Review(?): We have almost built the entire standard model Lagrangian. There are two remaining issues to the basic construction: fermion mixing and fermion masses (yes, we already put in fermion masses back in problem 7, but there is a problem with this now!). Let's deal with the fermion mixing issue in this problem. We'll look at the quark sector; the lepton sector, with massive neutrinos may be treated similarly (at least if the neutrinos are Dirac particles – they may well be Majorana particles instead, but either can mix).

We label the quark flavor eigenstates under the strong interaction as u, c, t, d, s, b. That is, these flavors are conserved in the strong interaction. According to the standard model, the left-handed quarks are arranged into weak isospin doublets and the right-handed quarks into weak isospin singlets. The $i_3 = +1/2$ states are, in general:

$$\begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L = V_+ \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L \tag{58}$$

and the $i_3 = -1/2$ states are:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = V_- \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L , \tag{59}$$

where V_+ and V_- are 3×3 unitary matrices. The charged current raising interaction Lagrangian is:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \begin{pmatrix} \bar{u}' & \bar{c}' & \bar{t}' \end{pmatrix} \frac{1}{2}(1-\gamma^5)\gamma^\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} W^+_\mu . \tag{60}$$

(a) Show that this Lagrangian is equivalent to the form:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \frac{1}{2}(1-\gamma^5)\gamma^\mu V \begin{pmatrix} d \\ s \\ b \end{pmatrix} W_\mu^+, \tag{61}$$

where V is determined by V_+ and V_-. Considering the fact that the quark fields are defined up to arbitrary phase conventions, determine how many essential real parameters are required to specify V. Show that, except for certain values of the parameters, this Lagrangian implies CP violation.

(b) Now consider the weak neutral current interaction. Show that there are no flavor-changing neutral current interactions. Show (via an example of a Feynman graph) that, with the charged W's, it is possible to have flavor-changing "neutral" current interactions in non-tree-level processes. For example, show how a b-quark can be turned into an s-quark. Carry this idea into a physically realizable diagram, for the process $B^0 \to K^0 e^+ e^-$. Never mind the details of the meson wave functions.

46. Of great current interest are measurements relevant to neutrino oscillations. Let us consider some of the relevant phenomenology, which has some similarities with our discussion of the neutral kaon system:

The phenomenon of neutrino oscillations can occur if the neutrino flavor eigenstates are mixtures of the neutrino mass eigenstates:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

where U is a unitary 3×3 matrix. Clearly, the neutrinos cannot all be degenerate in mass if non-trivial mixing occurs. In particular, at least one neutrino must have a non-zero mass.

(a) Suppose that only the first two generations mix, with mixing angle θ. Suppose further that we produce a muon anti-neutrino with momentum p, in the decay of an anti-muon at rest. Find the time-dependent probability that it will be observed as an electron neutrino if a measurement is made. Express your answer in terms of the mixing angle θ, and the difference in the square of the neutrino masses: $\Delta m^2 = m^2_{\nu_e} - m^2_{\nu_\mu}$. If you make any approximations, justify their use.

(b) The LSND experiment measured a $\bar{\nu}_\mu \to \bar{\nu}_e$ probability of $(0.31 \pm 0.12 \pm 0.05)\%$, using $\bar{\mu}$ decays at rest as the source of $\bar{\nu}_\mu$. The actual experiment involves a spread in momentum (because the $\bar{\mu} \to e^+ \nu_e \bar{\nu}_\mu$ decay is three-body), and a spread in distances between production and interaction points. However, we can get a pretty good idea for the numbers by using typical values. Thus, assume a $\bar{\nu}_\mu$ momentum of 50 MeV, and a path length of 30 m for LSND. Use the central value of the measured probability to obtain: (*i*) The lower bound on Δm^2 (assuming one of the masses is negligible, turn this into a bound on the higher mass neutrino); and (*ii*) The lower bound on $\sin^2 2\theta$. Of course, a proper error analysis is ultimately required, and gives allowed regions in the $\Delta m^2 - \sin^2 2\theta$ plane, as shown in the papers and seminars.

47. Time Reversal in Quantum Mechanics, Part II

We earlier showed that the time reversal operator, T, could be written in the form:

$$T = UK,$$

where K is the complex conjugation operator and U is a unitary operator. We also found that

$$T^2 = \pm 1.$$

Consider a spinless, structureless particle. All kinematic operators for such a particle may be written in terms of the $\vec{X}$ and $\vec{P}$ operators, where

$$[P_j, X_k] = -i\delta_{jk}$$

$$T\vec{X}T^{-1} = \vec{X}$$

$$T\vec{P}T^{-1} = -\vec{P}$$

(where the latter two equations follow simply from classical correspondence).

If we work in a basis consisting of the eigenvectors of $\vec{X}$, the eigenvalues are simply the real position vectors, and hence:

$$U\vec{X}U^{-1} = \vec{X}.$$

In this basis, the matrix elements of $\vec{P}$ may be evaluated:

$$\vec{P} = -i\vec{\nabla}:$$

$$\begin{aligned} \langle \vec{x}_1 \mid \vec{P} \mid \vec{x}_2 \rangle &= \int_{(\infty)} \delta^{(3)}(\vec{x} - \vec{x}_1)(-i\vec{\nabla}_x)\delta^{(3)}(\vec{x} - \vec{x}_2)d^{(3)}\vec{x} \\ &= -i\vec{\nabla}_{x_1}\delta^{(3)}(\vec{x}_1 - \vec{x}_2). \end{aligned}$$

Thus, these matrix elements are pure imaginary, and

$$K\vec{P}K^{-1} = -\vec{P},$$

which implies finally

$$U\vec{P}U^{-1} = \vec{P}.$$

We conclude that for our spinless, structureless particle:

$$U = 1e^{i\theta},$$

where the phase θ may be chosen to be zero if we wish. In any event, we have:

$$T = e^{i\theta}K,$$

and

$$T^2 = e^{i\theta}Ke^{i\theta}K = 1.$$

(a) Show that, for a spin 1/2 particle, we may in the Pauli representation write:

$$T = \sigma_2 K,$$

and hence show that:

$$T^2 = -1.$$

By considering a direct product space made up of many spin-0 and spin 1/2 states (or by other equivalent arguments), this result may be generalized: If the total spin is 1/2-integral, then $T^2 = -1$; otherwise $T^2 = +1$.

(b) Show the following useful general property of an antiunitary operator such as T:

Let

$$|\psi'\rangle = T|\psi\rangle$$
$$|\phi'\rangle = T|\phi\rangle.$$

Then

$$\langle\psi'|\phi'\rangle = \langle\phi|\psi\rangle.$$

This, of course, should agree nicely with your intuition about what time reversal should do to this kind of scalar product.

(c) Show that, if $|\psi\rangle$ is a state vector in an "odd" system ($T^2 = -1$), then $T|\psi\rangle$ is orthogonal to $|\psi\rangle$.

48. Let us think, in general terms, about some general scaling characteristics of cross sections in two different domains. We assume that we are dealing with a gauge theory. Let E be the center-of-mass energy.

(a) For any given process, what is the energy dependence of the cross section at very high energy? (Say what you mean by very high energy).

(b) Suppose the relevant gauge boson mass M is very large compared with E, and that E is large compared with the masses of the scattering particles. How does the cross section scale with E? State any additional assumptions you make.

(c) Does the scaling law in part b) encounter fundamental difficulty, outside of the details of our model, as E increases? At what energy does there become a problem?

49. In problem 38 you determined the Dalitz plot kinematic boundary for a 3 photon decay. Now find the kinematic boundary for the general case, of a particle of mass M decaying to three particles with masses m_1, m_2, and m_3. Pick some set of (m_i/M) $i = 1, 2, 3$ (different from problem 40) and make a graph in the $(m_{13}/M)^2, (m_{23}/M)^2$ plane showing what the boundary looks like. Note that the particle data booklet has a discussion of Dalitz plots. A classic reference on the analysis of Dalitz plots is: C. Zemach, Phys. Rev. **133** (1964) B1201.