Physics 231a
Problem Set Number 10
Due Wednesday, December 8 2004

Please turn in to my mail slot on the 3rd floor of Lauritsen by 4:00PM, Wednesday, December 8.

Note: Some problems may be "review" for some of you. I am deliberately including problems which are potentially in this category. If the material of the problem is already well-known to you, such that doing the problem would not be instructive, just write "been there, done that", or suitable equivalent, for that problem, and I'll give you credit.

50. Standard Model Review(?): We mentioned in problem 45 that there is a problem with our earlier introduction of the fermion mass terms in the standard model Lagrangian. Let us investigate this here.

    (a) Show that a fermion ($f$) mass term in the Lagrangian of the form:

    $$\mathcal{L}_{\text{mass}} = m\bar{f}f \tag{62}$$

    is not gauge invariant under $SU(2)_L$. Thus, we have a problem, since we know experimentally that (most of) the fermions are not massless.

    (b) We'll solve this problem with the Higgs, just as we did for the gauge bosons. Neglecting mixing now, consider the $u_L, d_L$ doublet and $u_R, d_R$ singlets under $SU(2)_L$. Suppose we have a term in the Lagrangian of the form

    $$\begin{aligned} \mathcal{L}_{\bar{f}\phi f} &= -\lambda_d(\bar{u}_L, \bar{d}_L)\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix} d_R \\ &\quad -\lambda_u(\bar{u}_L, \bar{d}_L)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\phi_1 - i\phi_2\\ \phi_3 - i\phi_4\end{pmatrix} u_R \\ &\quad +\text{h.c.}, \end{aligned} \tag{63}$$

    where $L$ and $R$ refer to left- and right-handed, respectively, $\lambda_{d,u}$ are real constants, and $\phi = \begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}$ is a complex $SU(2)_L$ doublet scalar field. Show that $\mathcal{L}_{\bar{f}\phi f}$ is gauge invariant.

(c) Let the scalar doublet acquire a vacuum expectation value:

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}. \tag{64}$$

Show that the fermions acquire masses, and evaluate the masses in terms of the constants in this problem.

Note that this explains the standard model prediction that the Higgs particle couples to fermions with coupling proportional to the fermion mass.

51. Time reversal and the Dirac Equation: We have already argued, from ordinary quantum mechanics, that the time reversal operator $T$ could be written as $T = UK$, where $U$ is a unitary matrix, and $K$ is the complex conjugation operator. This result was based on the commutation relations among kinematic operators, and classical correspondence, so we expect a similar result to obtain in the Dirac theory. The time reversal transformation of our coordinates is $x' = \Lambda x$, with:

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Under time reversal, we expect our spinor function $\psi$ to transform according to

$$\begin{aligned}\psi(x) \to \psi'(x') &= S_T\psi(x)\\ &= V_T\psi^*(x),\end{aligned}$$

where we assume $S_T = V_T K$, and $V_T$ is a linear operator.

(a) By demanding that the Dirac equation be invariant under time reversal, determine $V_T$ in the Dirac-Pauli representation. Your discussion will likely be analogous to our discussion in class for parity.

(b) Carefully check that your result for $S_T$ is antiunitary (if it isn't, better fix it !), and show that $(S_T)^2$ is what you expect given that the Dirac equation is supposed to be a theory of spin $\frac{1}{2}$ particles. Using your explicit Dirac-Pauli form, argue that $S_T$ does an intuitively plausible thing when acting on the plane wave spinors $\psi(x)$.

52. In class we have introduced the notion of a form factor, as a modification to a point-like cross section for scattering on an extended object. We have stated that the form factor for the scattering on a static, spinless charge distribution is given by the Fourier transform of the charge distribution. The purpose of this problem is to demonstrate this.

    (a) Starting with the expression for the transition amplitude in terms of a current interacting with an electromagnetic field:

    $$T_{fi} = -i \int d^4(x) j^{\mu}_{fi}(x) A_{\mu}(x),$$

    show that the cross section may be written in the form:

    $$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} F(\mathbf{q}^2),$$

    where $F(\mathbf{q}^2)$ is the Fourier transform of the target charge distribution.

    (b) The scattering of an electron on a static spinless point target is called Mott scattering. Show that, in the limit $v \to 0$, the Mott cross section approaches the Rutherford cross section for the Coulomb scattering of a spinless projectile on a static spinless target. Why?

53. Let's think about some of the useful kinematic variables introduced in class, in our discussion of deep inelastic scattering. Recall, in lepton-hadron scattering:

    $$-Q^2 \equiv q^2 = (k - k')^2,$$

    where $k$ and $k'$ are the initial and final lepton momenta, respectively. Also,

    $$\nu \equiv \frac{p \cdot q}{M},$$

    where $p$ is the initial hadron momentum, and $M$ is its mass. The pair $(Q^2, \nu)$ is a suitable choice of kinematic invariants for describing the scattering. However, we could also choose the dimensionless pair $(x, y)$, where:

    $$x \equiv \frac{Q^2}{2p \cdot q}, \qquad y \equiv \frac{p \cdot q}{p \cdot k}.$$

(a) Determine the kinematically allowed regime in $x$ and $y$, for $\mu p \to \mu X$ scattering. You may use the relativistic limit as desired.

(b) Make a graph of the mapping of lines of constant $x$ and of constant $y$ on the $(Q^2, \nu)$ plane.

54. The resonant scattering of two particles, $a$ and $b$, through a resonance of spin $J$ can be described by the formula (see, *e.g.*, Perkins, section 4.8):

$$\sigma(E) = \frac{2J+1}{(2S_a+1)(2S_b+1)} \frac{4\pi}{E^2} \frac{\Gamma_{ab}\Gamma_{\text{fin}}}{(M-E)^2+\Gamma^2/4}.$$

The idea here is the reaction:

$$a + b \to M \to \quad \text{final state},$$

where M is the resonance mass as well as the label for the resonance. $S_a$ and $S_b$ are the spins of the colliding particles, the factor $\frac{1}{(2S_a+1)(2S_b+1)}$ averages over the initial spin states. The $2J+1$ factor sums over the possible intermediate resonance spin states. $\Gamma$ is the total width of our Breit-Wigner resonance, $\Gamma_{ab}$ is its decay rate to particles $a$ and $b$, and $\Gamma_{\text{fin}}$ is its decay rate to the specified final state:

Consider the case of a heavy quarkonium resonance, such as $J/\psi$ or $\Upsilon$, produced in $e^+e^-$ annihilation. In this case, practical limitations are such that the energy spread of the colliding beam machine ($\gtrsim$1 MeV) is much greater than the width of the resonance ($\lesssim$200 keV). It is reasonable to assume that the machine energy distribution is Gaussian:

$$\varrho(E; E_0) = \frac{1}{\sqrt{2\pi}\delta_E} \exp\left[-\frac{1}{2}\frac{(E-E_0)^2}{\delta_E^2}\right]$$

(a) Obtain, as a function of the mean machine energy $E_0$, an expression for the observed cross section, $\sigma_{e^+e^-\to M\to \text{final state}}(E_0)$. Show that a measurement of the cross section for $e^+e^- \to M \to e^+e^-$ at the peak of the resonance yields $B_{ee}\Gamma_{ee}$, the branching fraction for resonance decay to $e^+e^-$ times the partial width into $e^+e^-$. Likewise, the partial width to $e^+e^-$ can be determined by measuring the cross section for $e^+e^- \to$ M $\to$ anything at the peak. Hence, $B_{ee}$ can be determined. (Don't worry about radiative corrections in this problem).

(b) Knowing $B_{ee}$ from part (a), show that a measurement of the area under the observed resonance peak to any specified final state yields the decay width of the resonance to that final state. (In particular, we may obtain the total width – or lifetime – of the resonance in this manner).