Ph196bEM Final Exam Solutions

Mar 24, 2004

■ Problem 1.

1) (10 points) A commonly used particle detector consists of a coaxial arrangement of two long cylindrical conductors with a potential difference between them, and with a gas in the contained volume. Ionizing radiation creates electron-ion pairs in the gas and these separate, with the positive ion moving toward the negative electrode and the electron, toward the positive one. As the charges move, the amount of charge induced by them on the conductors changes, giving rise to a current in the external circuit which maintains the potential between the conductors; it is this current that is detected, signalling the ionizing event in the detector.

Suppose the two conducting cylinders have length L, and radii $a < b \ll L$, and that an external battery maintains a potential V between them, the center conductor being positive. If a point charge q is at radius ρ, $a < \rho < b$, and it is far from the ends of the detector, what is the induced charge on each of the conductors? (Ignore the dielectric constant of the gas.)

Note: If you knew the mobilities of electrons and ions (*mobility* is the ratio of velocity to electric field) in the gas you could work out the current in the external circuit with the results 0f this problem, but you needn't do that for this exam.

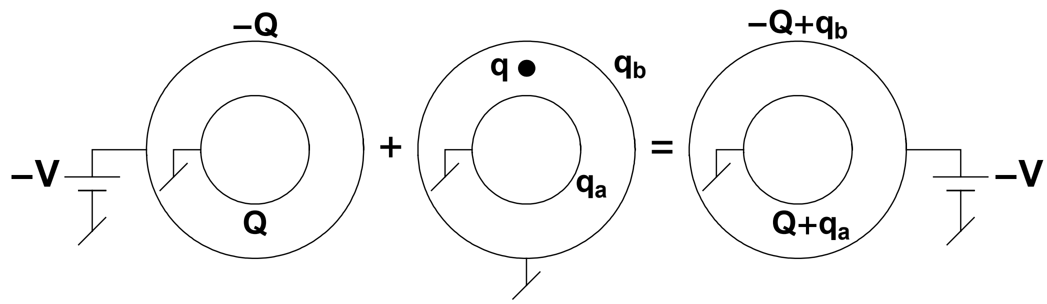
■ Solution 1.

It is convenient (but not necessary) to arrange things as much as possible in a problem in a problem is at ground potential. Since we only want induced charges and no potentials in this problem, notice that for this calculation, all the conductors can be grounded. Use superposition to get rid of the potential V – the <u>induced</u> charge on the conductors when there is a potential V between them is the same as the induced charge on them when both are grounded.

Begin graphics

end graphics

In[2]:= `Show@super1`



Out[2]= `- Graphics -`

I – Solving by Green's Reciprocity Theorem

Use Green's reciprocity theorem on the following two situations

Begin graphics

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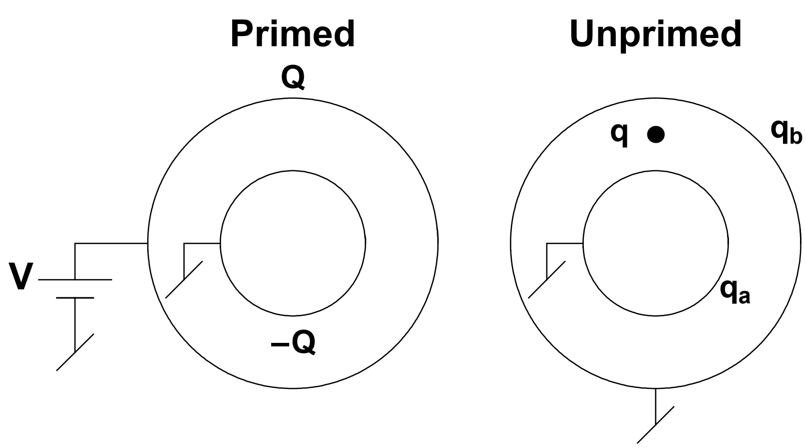
In[4]:= `Show@green`



Out[4]= `- Graphics -`

So we get

$$q\,\Phi'(\vec{\rho}_q) + \sum_i q_i\,\Phi'_i = \sum_i q'_i\,\Phi_i \;\Rightarrow\; q\,\Phi'(\vec{r}_q) + q_b\,V + q_a \times 0 = Q \times 0 - Q \times 0 = 0 \quad \Rightarrow \quad q_b = -q\,\frac{\Phi'(\vec{r}_q)}{V}$$

In the primed case, we have from symmetry and Gauss's law $2\,\pi\,\rho\,L\,E_\rho\,(\rho) = -\frac{1}{\varepsilon_0}\,Q \Rightarrow$
$\Phi(\rho) = \Phi_0 + \frac{1}{2\,\pi\,\varepsilon_0}\,\frac{Q}{L}\,\log\rho$.

We need to adjust constants so that $\Phi(a) = 0$, and $\Phi(b) = V$ so it is easy to see that

$$\Phi(\rho) = V \,\frac{\log(\frac{\rho}{a})}{\log(\frac{b}{a})}$$

and so we find for the induced charge on the outer cylinder

$$q_b = -q\,\frac{\log(\frac{\rho}{a})}{\log(\frac{b}{a})}.$$

Since an integral of $\vec{E}$ over a surface contained inside the material of the outer conductor gives zero (since $\vec{E}$ =0), we must have

$$q_a + q + q_b = 0 \qquad \Rightarrow \qquad q_a = -(q + q_b) = -q\,\frac{\log(\frac{b}{\rho})}{\log(\frac{b}{a})}.$$

II – Solving using a two dimensional Green function

Begin by arguing that, for a point charge between the two electrodes, the amount of induced charge is independent of the z position of the charge. Thus if we find the amount of charge λ' induced by a given line charge λ, then the amount of point charge q' induced by q must satisfy

$$\frac{q'}{q} = \frac{\lambda'}{\lambda}.$$

The Green function for a unit line charge $\frac{1}{\rho}\,\delta(\rho-\rho_0)\,\delta(\varphi)$ between two infinitely-long grounded cylinders of radii a and b, where $a < \rho_0 < b$, is

$$G = A_0\, g_0(\rho) + \sum_{m=1}^{\infty} A_m \cos m\varphi\, g_m(\rho)$$

where for $m > 0$

$$g_m(\rho) = \begin{pmatrix} ((\frac{\rho}{a})^m - (\frac{\rho}{a})^{-m})((\frac{\rho_0}{b})^m - (\frac{\rho_0}{b})^{-m}) & a < \rho < \rho_0 \\ ((\frac{\rho_0}{a})^m - (\frac{\rho_0}{a})^{-m})((\frac{\rho}{b})^m - (\frac{\rho}{b})^{-m}) & \rho_0 < \rho < b \end{pmatrix}$$

and

$$g_0(\rho) = \begin{pmatrix} \log\frac{\rho}{a}\,\log\frac{\rho_0}{b} & a < \rho < \rho_0 \\ \log\frac{\rho_0}{a}\,\log\frac{\rho}{b} & \rho_0 < \rho < b \end{pmatrix}$$

The charge on a unit length of the outer cylinder of radius b is

$$\lambda_b = \varepsilon_0 \int_{-\pi}^{\pi} \partial_\rho G \,\Big|_{\rho=b}\, b\, d\varphi = 2\,\pi\,\varepsilon_0\, A_0 \log\frac{\rho_0}{a}$$

So we only need the constant A_0.

To get it, integrate the equation

$$\nabla^2 G = -\frac{1}{\varepsilon_0\,\rho}\,\delta(\rho-\rho_0)\,\delta(\varphi)$$

over all φ to get

$$2\,\pi\,A_0\,\frac{1}{\rho}\,(\rho\,g_0'(\rho))' = -\frac{1}{\varepsilon_0\,\rho}\,\delta(\rho-\rho_0)$$

and then over the singularity in ρ giving

$$2\,\pi\,A_0\,\rho_0\left(g_0'(\rho)\,\Big|_{\rho=\rho_0+\epsilon} - g_0'(\rho)\,\Big|_{\rho=\rho_0-\epsilon}\right)$$
$$= 2\,\pi\,A_0\,\rho_0\left(\log\left(\frac{\rho_0}{a}\right)\frac{1}{\rho_0} - \log\left(\frac{\rho_0}{b}\right)\frac{1}{\rho_0}\right)$$
$$= 2\,\pi\,A_0\,\log\left(\frac{b}{a}\right)$$
$$= -\frac{1}{\varepsilon_0}.$$

So since $q_b = q\,\lambda_b$ (since $\lambda = 1$) we finally get

$$q_b = q\,2\,\pi\,\varepsilon_0\,A_0\,\log\frac{\rho_0}{a} = -q\,\frac{\log\frac{\rho_0}{a}}{\log\left(\frac{b}{a}\right)}$$

as we got earlier.

III – Solving with 3D Green's Function with non-oscillatory function along *z*

Another approach, NOT the one I expected people to use, is to go to the trouble of solving the full electrostatic problem for the case of a point charge q in the space between two grounded cylinders (the 3D Green's function problem). Note that this is not a 2D electrostatics problem because of the z dependence created by the point charge.

Put the point charge at $(\rho, \varphi, z) = (\rho_0, 0, 0)$ for convenience. Since:

a) the potential must go to zero at infinity,

b) we are using real exponentials in the z direction,

c) the potential must be single valued and symmetric under $\varphi \to -\varphi$ so we must have a φ dependence of $\cos m\varphi$ where $m = 0, 1, 2, 3, \ldots$,

d) the origin ρ is not in the volume of interest, we have for the ρ dependent functions a linear combination of the two Bessel functions $J_m(k\rho)$ and $Y_m(k\rho)$.[Note that Jackson calls the oscillatory Bessel functions which are not regular at the origin $N_m(k\rho)$, but *Mathematica* uses $Y_m(k\rho)$. I adopt that convention to make evaluation easy.] The linear combination and the values of k must be chosen to get zero potentials at both $\rho = a$ and $\rho = b$.

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}(J_m(k_{mn}a)\,Y_m(k_{mn}\rho) - J_m(k_{mn}\rho)\,Y_m(k_{mn}a))\cos m\varphi\, e^{-k_{mn}|z|}$$

where k_{mn} is chosen so that $J_m(k_{mn}a)\,Y_m(k_{mn}b) - J_m(k_{mn}b)\,Y_m(k_{mn}a) = 0$, and the coefficients A_{mn} are chosen so Poisson's equation is satisfied. The index n counts the solutions of $J_m(k_{mn}a)\,Y_m(k_{mn}b) - J_m(k_{mn}b)\,Y_m(k_{mn}a) = 0$. For convenience, let

$f_m(k_{mn}\rho) = J_m(k_{mn}a)\, Y_m(k_{mn}\rho) - J_m(k_{mn}\rho)\, Y_m(k_{mn}a)$.

We need $\nabla^2 \Phi = -\frac{q}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho_0)\, \delta(\varphi)\, \delta(z)$ so

$$\nabla^2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, f_m(k_{mn}\rho) \cos m\varphi\, e^{-k_{mn}|z|}$$
$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, f_m(k_{mn}\rho) \cos m\varphi\, (g'' - k_{mn}^2\, g)$$
$$= -\frac{q}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho_0)\, \delta(\varphi)\, \delta(z) \qquad \text{eq. 1}$$

where $g = e^{-k_{mn}|z|}$, the function with a discontinuous slope.

Because both the Bessel functions $J_m(k_{mn}\rho)$ and $Y_m(k_{mn}\rho)$ satisfy the Stürm–Liouville equation with the positive function "$p(x)$" = ρ and the other function "$q_i(x)$" = $k_{mn}^2 \rho - \frac{m^2}{\rho}$, and so also does their linear combination $f_{mn}(\rho)$, you get the orthogonality relation

$$\int_a^b (k_{mn}^2 - k_{mn'}^2) \rho\, f_m(k_{mn}\rho)\, f_m(k_{mn'}\rho)\, d\rho =$$
$$b(f_m(k_{mn}b)\, f_m'(k_{mn'}b) - f_m'(k_{mn}b)\, f_m(k_{mn'}b)) - a(f_m(k_{mn}a)\, f_m'(k_{mn'}a) - f_m'(k_{mn}a)\, f_m(k_{mn'}a)) = 0,$$

for $n \neq n'$. This is zero because we have constructed the functions $f_m(k_{mn}\rho)$ and $f_m(k_{mn'}\rho)$ so that they both vanish at both $\rho = a$ and $\rho = b$. For convenience again, let

$\int_a^b \rho\, f_m^2(k_{mn}\rho)\, d\rho = C_{mn}$ so that

$\int_a^b \rho\, f_m(k_{mn}\rho)\, f_m(k_{mn'}\rho)\, d\rho = \delta_{nn'}\, C_{mn}$.

We also have the orthogonality relation in the azimuthal angle:

$$\int_0^{2\pi} \cos m\varphi \cos m'\varphi\, d\varphi = \begin{pmatrix} 0 & m \neq m' \\ \pi & m > 0 \text{ and } m = m' \\ 2\pi & m = 0 \text{ and } m' = 0 \end{pmatrix}$$
$$= (1 + \delta_{m0})\, \pi\, \delta_{mm'}.$$

So multiply both sides of eq. 1 with $\rho\, f_{m'}(k_{m'n'}\rho) \cos m'\varphi$ and integrate over ρ from a to b and over φ from 0 to 2π to get

$$A_{m'n'}\, C_{m'n'} (1 + \delta_{m'0})\, \pi\, (g'' - k_{m'n'}^2\, g) = -\frac{q}{\varepsilon_0} f_{m'}(k_{m'n'}\, \rho_0)\, \delta(z)$$

Finally integrate over z from $-\epsilon$ to $+\epsilon$, $\epsilon > 0$, and take the limit as $\epsilon \to 0$ to get

$$A_{m'n'}\, C_{m'n'} (1 + \delta_{m'0})\, \pi \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \left(g' \Big|_{z=\epsilon} - g' \Big|_{z=-\epsilon} \right)$$
$$= -2\, k_{m'n'}\, A_{m'n'}\, C_{m'n'} (1 + \delta_{m'0})\, \pi$$
$$= -\frac{q}{\varepsilon_0} f_{m'}(k_{m'n'}\, \rho_0).$$

So we get

$$A_{mn} = \frac{q}{2\,(1 + \delta_{m0})\, \pi\, \varepsilon_0} \frac{f_m(k_{mn}\, \rho_0)}{k_{mn}\, C_{mn}}.$$

The next step is to get the charge density on the cylinder at $\rho = a$ integrate it over φ and z to get q_a.

$$q_a = \varepsilon_0 \int_{\varphi=0}^{2\pi} \int_{z=-\infty}^{\infty} \left(-\frac{\partial \Phi}{\partial \rho}\right)_{\rho=a} a\, d\varphi\, dz.$$

Finally then an opaque answer for the induced charge is

$$q_a = -2\,\varepsilon_0 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, k_{mn}\, a\, (J_m(k_{mn}a)\, Y_m'(k_{mn}a) - J_m'(k_{mn}a)\, Y_m(k_{mn}a)) \left(\int_0^{2\pi} \cos m\varphi\, d\varphi\right) \left(\int_0^{\infty} e^{-k_{mn} z}\, dz\right)$$

or

$$\boxed{q_a = -q \sum_{n=1}^{\infty} \frac{1}{k_{0n}\, C_{0n}}\, a\, (J_0(k_{0n}a)\, Y_0'(k_{0n}a) - J_0'(k_{0n}a)\, Y_0(k_{0n}a))\, (J_0(k_{0n}a)\, Y_0(k_{0n}\rho_0) - J_0(k_{0n}\rho_0)\, Y_0(k_{0n}a))}$$

Since $J_0(k\,\rho)$ and $Y_0(k\,\rho)$ satisfy the *same* Stürm–Liouville equation (with $p(x) = x$) we have $x\,(J_0(x)\, Y_0'(x) - J_0'(x)\, Y_0(x)) = \text{cnst}$. The value of the constant depends on the normalization of the Bessel functions, so to conform with *Mathematica* conventions, calculate

```
In[5]:= FullSimplify[
          x (BesselJ[0, x] (∂x BesselY[0, x]) - BesselY[0, x] (∂x BesselJ[0, x]))]
```

Out[5]= $\frac{2}{\pi}$

Thus we get

$$q_a = -q\, \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{k_{0n}^2\, C_{0n}}\, (J_0(k_{0n}a)\, Y_0(k_{0n}\rho_0) - J_0(k_{0n}\rho_0)\, Y_0(k_{0n}a))$$

Numerical comparisons between the series solution and the simple ratio of logs

To make the comparison, we need to have the values of C_{0n}. It,as well as the derivatives, can be evaluated with *Mathematica*, but it does a poor job on the C_{0n}; I'll do it by hand.

Use Abramowitz and Stegun, 11.4.2.

$$C_{0n} = \frac{k_{0n}^2}{2} \left(b^2\, (J_0(k_{0n}a)\, Y_0'(k_{0n}b) - J_0'(k_{0n}b)\, Y_0(k_{0n}a))^2 - a^2\, (J_0(k_{0n}a)\, Y_0'(k_{0n}a) - J_0'(k_{0n}a)\, Y_0(k_{0n}a))^2\right)$$

Find the derivatives.

```
In[6]:= ∂x { BesselJ[0, x], BesselY[0, x]}
Out[6]= {-BesselJ[1, x], -BesselY[1, x]}
```

$$\boxed{C_{0n} = \frac{k_{0n}^2}{2} \left(b^2\, (J_0(k_{0n}a)\, Y_1(k_{0n}b) - J_1(k_{0n}b)\, Y_0(k_{0n}a))^2 - a^2\, (J_0(k_{0n}a)\, Y_1(k_{0n}a) - J_1(k_{0n}a)\, Y_0(k_{0n}a))^2\right)}$$

Do a numerical check of this relation. But first we need the zeros of the cylinder function.

```
In[7]:= << NumericalMath`BesselZeros`
```

BesselJYJYZeros[0, λ, num] is designed to give the first *num* zeros of $J_0(x)\,Y_0(\lambda\, x) - J_0(\lambda\, x)\,Y_0(x)$. We want the values of $k_{0\,n}$ so that $J_0(k_{0n} a)\,Y_0(k_{0n} b) - J_0(k_{0n} b)\,Y_0(k_{0n} a) = 0$, so let the scale be set by a, i.e., $a = 1$, and x is then k_{0n} and $\lambda = b$, the outer radius. I choose $b/a = 2$.

```
In[48]:= k0n = BesselJYJYZeros[0, 2., 200];
```

Check that these k values make the outer cylinder zero potential.

```
In[9]:= fcylinder[k_, ρ_] :=
         BesselJ[0, k ] BesselY[0, k ρ ] - BesselY[0, k ] BesselJ[0, k ρ ]

In[10]:= roots = {1, 10, 30, 50, 100};

In[11]:= fcylinder[k0n[[#]], 1] & /@ roots

Out[11]= {0., 0., 0., 0., 0.}
```

The inner cylinder is grounded.

```
In[12]:= fcylinder[k0n[[#]], 2] & /@ roots
```

Out[12]= $\{-1.08247\times10^{-15},\ 8.67362\times10^{-19},\ 2.38524\times10^{-17},\ -2.29851\times10^{-17},\ 1.74557\times10^{-17}\}$

And so is the outer one. So the values of k_{0n} look good. Now do numerical integrations to get C_{0n} for some values of n in order to check the hand evaluation of this integral.

```
In[13]:= numint = NIntegrate[ρ fcylinder[k0n[[#]], ρ]^2, {ρ, 1, 2}] & /@ roots;

In[14]:= formula =
          (1/2 (2^2 (BesselJ[0, k0n[[#]]] BesselY[1, 2 k0n[[#]]] - BesselJ[1, 2 k0n[[#]]] BesselY[
                    0, k0n[[#]]])^2 - (BesselJ[0, k0n[[#]]] BesselY[1, k0n[[#]]] -
                BesselJ[1, k0n[[#]]] BesselY[0, k0n[[#]]])^2)) & /@ roots;
```

Now compare the two evaluations.

```
In[15]:= Transpose[{numint, formula}] // TableForm
```

Out[15]//TableForm=

0.0204364	0.0204364
0.000205307	0.000205307
0.0000228131	0.0000228131
8.21277×10^{-6}	8.21277×10^{-6}
2.0532×10^{-6}	2.0532×10^{-6}

The comparison between the hand calculation and the numerical evaluation of the integral looks good, so I have confidence that the formula is OK. Now let's check if the long ugly series gives the same result as does the elegant Green's reciprocity theorem. Evaluate $r = -q_a/q$ using $b/a = 2$ and values of ρ_0/a from 1.1 to 1.9.

In the table below, the first column gives the ρ_0 / a value, the second comes from numerical evaluation of the series, and the third column, from the ratio of logs.

```
In[49]:= Module[{b = 2., ρ0 = #}, {ρ0, rseries =
      -Sum[Module[{kn = k0n[[n]], cn}, cn = (1/2 (b^2 (BesselJ[0, kn] BesselY[1, b kn] -
                 BesselJ[1, b kn] BesselY[0, kn])^2 - (BesselJ[0, kn]
                 BesselY[1, kn] - BesselJ[1, kn] BesselY[0, kn])^2));
           1/(kn^2 cn) 2/π (BesselJ[0, kn ρ0] BesselY[0, kn] -
              BesselJ[0, kn ] BesselY[0, kn ρ0])], {n, 1, 200}],
      rgreenrecip = Log[b / ρ0] / Log[b]}] & /@ Table[ρ0, {ρ0,
     1.1, 1.9, .1}] // TableForm
```

Out[49]//TableForm=

1.1	0.85292	0.862496
1.2	0.732495	0.736966
1.3	0.618749	0.621488
1.4	0.512722	0.514573
1.5	0.413738	0.415037
1.6	0.321014	0.321928
1.7	0.233843	0.234465
1.8	0.151618	0.152003
1.9	0.0738177	0.0740006

This shows that the series is reasonable, but because the result is a log, the convergence is slow. Even 200 terms does not give high accuracy.

IV– Solving with 3D Green's Function with non-oscillatory function along ρ

Again, this is not the way I expected this problem to be attacked, but if one chooses to use oscillatory solutions in the z direction to solve the full electrostatics problem, then there is no restriction on the separation constant k. So we have a Fourier integral along z and a Fourier series in φ. The potential can be represented as

$$\Phi = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, A_m(k) \cos m\varphi\, \cos kz\, g_m(k\rho).$$

I have chosen cosines because of the symmetry under $\varphi \to -\varphi$ and $z \to -z$. Similarly, I only need positive values of m and k. The function $g_m(k\rho)$ is constructed out of modified Bessel functions: (I_m is regular at the origin and K_m is singular there)

$$g_m(k\rho) = \begin{pmatrix} (I_m(k\rho) K_m(ka) - I_m(ka) K_m(k\rho))(I_m(k\rho_0) K_m(kb) - I_m(kb) K_m(k\rho_0)), & a \le \rho < \rho_0 \\ (I_m(k\rho_0) K_m(ka) - I_m(ka) K_m(k\rho_0))(I_m(k\rho) K_m(kb) - I_m(kb) K_m(k\rho)), & \rho_0 < \rho \le b \end{pmatrix}$$

The function is constructed to be zero at both $\rho = a$ and $\rho = b$, and to be continuous at $\rho = \rho_0$.

Now we choose $A_m(k)$ so that $\nabla^2 \Phi = -\frac{q}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi) \delta(z)$ or, with a slight notation change,

$$\sum_{m'=0}^{\infty}\int_0^{\infty} dk'\, A_{m'}(k') \cos m'\varphi \cos k'z \left(\frac{1}{\rho}\, \frac{d}{d\rho} \left(\rho\, \frac{d g_{m'}(k'\,\rho)}{d\rho}\right) - \left(\frac{m'^2}{\rho^2} + k'^2\right) g_{m'}(k'\,\rho)\right) = -\frac{q}{\varepsilon_0}\, \frac{1}{\rho}\, \delta(\rho-\rho_0)\, \delta(\varphi)\, \delta(z)$$

Multiply both sides by $\cos m\varphi \cos kz$, $m \geq 0$ and $k \geq 0$, and integrate over φ from $-\pi$ to π and over z from $-\infty$ to ∞. Use the relations

$$\int_{-\pi}^{\pi} \cos m\varphi \cos m'\varphi\, d\varphi = (1+\delta_{m0}\, \delta_{m'0})\, \pi\, \delta_{mm'}$$

and,

$$\int_{-\infty}^{\infty} \cos k\, z \cos k'\, z\, dz = \frac{1}{4}\, 2\, \pi\, (\, \delta(k+k') + \delta(k-k') + \delta(-k+k') + \delta(-k-k')\,)$$

$$= \pi\, (\delta(k+k') + \delta(k-k')) = \begin{pmatrix} \pi\, \delta(k-k') & k \neq 0 \\ 2\, \pi\, \delta(k') & k = 0 \end{pmatrix} = t(k)\, \pi\, \delta(k-k') \text{ where } t(k) = \begin{pmatrix} 1 & k > 0 \\ 2 & k = 0 \end{pmatrix}$$

to dissolve the sum and integral. This then gives

$$\sum_{m'=0}^{\infty}\int_0^{\infty} dk'\, A_{m'}(k')\, (1+\delta_{m0}\, \delta_{m'0})\, \pi\, \delta_{mm'}\, \pi\, t(k)\, \pi\, \delta(k-k') \left(\frac{1}{\rho}\, \frac{d}{d\rho} \left(\rho\, \frac{d g_{m'}(k'\,\rho)}{d\rho}\right) - \left(\frac{m'^2}{\rho^2} + k'^2\right) g_{m'}(k'\,\rho)\right) =$$
$$-\frac{q}{\varepsilon_0}\, \frac{1}{\rho}\, \delta(\rho-\rho_0).$$

The sum over m' and integral over k' are easy, yielding

$$A_m(k)\, (1+\delta_{m0})\, t(k)\, \pi^2 \left(\frac{1}{\rho}\, \frac{d}{d\rho} \left(\rho\, \frac{d g_m(k\,\rho)}{d\rho}\right) - \left(\frac{m^2}{\rho^2} + k^2\right) g_m(k\,\rho)\right) = -\frac{q}{\varepsilon_0}\, \frac{1}{\rho}\, \delta(\rho-\rho_0).$$

Multiply by ρ and integrate over ρ from $\rho_0 - \epsilon$ to $\rho_0 + \epsilon$, $\epsilon > 0$, and take the limit $\epsilon \to 0$ to get

$$A_m(k)\, (1+\delta_{m0})\, \pi^2\, \rho_0 \left(\frac{d g_m(k\,\rho)}{d\rho}\Bigg|_{\rho=\rho_0+\epsilon} - \frac{d g_m(k\,\rho)}{d\rho}\Bigg|_{\rho=\rho_0-\epsilon} \right) = -\frac{q}{\varepsilon_0}$$

From the definition of $g_m(k\,\rho)$ this gives

$$A_m(k)\, t(k)\, (1+\delta_{m0})\, \pi^2\, k\, \rho_0\, \mathcal{B}_m(k) = -\frac{q}{\varepsilon_0} \quad \Rightarrow \quad A_m(k) = -\frac{q}{t(k)\, (1+\delta_{m0})\, \pi^2\, \varepsilon_0}\, \frac{1}{k\, \rho_0\, \mathcal{B}_m(k)}$$

$$\mathcal{B}_m(k) = (\, (\, I_m(k\rho_0)\, K_m(ka) - I_m(ka)\, K_m(k\rho_0)\,)\, (\, I'_m(k\rho_0)\, K_m(kb) - I_m(kb)\, K'_m(k\rho_0)\,) -$$
$$(\, I'_m(k\rho_0)\, K_m(ka) - I_m(ka)\, K'_m(k\rho_0)\,)\, (\, I_m(k\rho_0)\, K_m(kb) - I_m(kb)\, K_m(k\rho_0)\,)\,).$$

Since both $(\, I_m(k\rho)\, K_m(ka) - I_m(ka)\, K_m(k\rho)\,)$ and $(\, I_m(k\rho)\, K_m(kb) - I_m(kb)\, K_m(k\rho)\,)$ satisfy the same Stürm–Liouville equation (with $p(x) = x$) in ρ we get $\mathcal{B}_m(k) = \frac{\text{cnst}}{k\,\rho_0}$. To get the constant, which depends on normalization conventions, use *Mathematica:*

```
In[50]:= FullSimplify[(BesselI[m, x] BesselK[m, k a] - BesselI[m, k a] BesselK[m, x])
           ∂x (BesselI[m, x] BesselK[m, k b] - BesselI[m, k b] BesselK[m, x])
          - ∂x (BesselI[m, x] BesselK[m, k a] - BesselI[m, k a] BesselK[m, x])
           (BesselI[m, x] BesselK[m, k b] - BesselI[m, k b] BesselK[m, x])]

Out[50]= 1/x (BesselI[m, b k] BesselK[m, a k] - BesselI[m, a k] BesselK[m, b k])
```

Thus we get the simpler form (not necessary, but nice)

$$\mathcal{B}_m(k) = \frac{I_m(kb)\, K_m(ka) - I_m(ka)\, K_m(kb)}{k\,\rho_0}.$$

To get the induced charge on the inner cylinder we need

$$q_a = \varepsilon_0 \int_{\varphi=0}^{2\pi} \int_{z=-\infty}^{\infty} \left(-\frac{\partial \Phi}{\partial \rho}\right)_{\rho=a} a \, d\varphi \, dz$$

So

$$q_a = \int_{\varphi=0}^{2\pi} \int_{z=-\infty}^{\infty} \sum_{m=0}^{\infty} \int_0^{\infty} dk \, \frac{q}{\iota(k)\,(1+\delta_{m0})\,\pi^2} \; \frac{1}{\rho_0 \, \mathcal{B}_m(k)} \cos m\varphi \cos kz \, g_m'(ka) \, a \, d\varphi \, dz$$
$$= \int_{z=-\infty}^{\infty} \int_0^{\infty} dk \, \frac{q}{2\pi^2} \; \frac{2\pi}{\iota(k)\,\rho_0\,\mathcal{B}_0(k)} \cos kz \, g_0'(ka) \, a \, dz$$
$$= \int_0^{\infty} dk \, \frac{q}{2\pi^2} \; \frac{(2\pi)^2}{\iota(k)\,\rho_0\,\mathcal{B}_0(k)} \, \delta(k) \, g_0'(ka) \, a$$
$$= q \, \frac{a}{\rho_0} \lim_{k\to 0} \frac{g_0'(ka)}{\mathcal{B}_0(k)}$$
$$= q \, \frac{a}{\rho_0} \lim_{k\to 0} \left(I_0'(ka)\, K_0(ka) - I_0(ka)\, K_0'(ka) \right) \frac{\left(I_0(k\rho_0)\, K_0(kb) - I_0(kb)\, K_0(k\rho_0) \right)}{\mathcal{B}_0(k)}$$

I'll not bother with the Wronskian this time.

```
In[17]:= ∂x {BesselI[0, x], BesselK[0, x]}

Out[17]= {BesselI[1, x], -BesselK[1, x]}
```

So using these evaluations of the derivatives, you get

$$q_a = q \, \frac{a}{\rho_0} \lim_{k\to 0} \left(I_1(ka)\, K_0(ka) + I_0(ka)\, K_1(ka) \right) \frac{\left(I_0(k\rho_0)\, K_0(kb) - I_0(kb)\, K_0(k\rho_0) \right)}{\mathcal{B}_0(k)}$$

where

$$\mathcal{B}_0(k) = \frac{I_0(kb)\, K_0(ka) - I_0(ka)\, K_0(kb)}{k\,\rho_0}$$

Use *Mathematica* to take the limit.

```
In[54]:= B0k = 1/(k ρ0) (BesselI[0, b k] BesselK[0, a k] - BesselI[0, a k] BesselK[0, b k]);

In[55]:= Limit[
           1/B0k ((BesselI[1, k a] BesselK[0, k a] + BesselI[0, k a] BesselK[1, k a])
              (BesselI[0, k ρ0] BesselK[0, k b] -
                BesselI[0, k b] BesselK[0, k ρ0])), k → 0] // FullSimplify

Out[55]= ρ0 (Log[b] - Log[ρ0])
         ---------------------
          a (Log[a] - Log[b])
```

So finally you get a nice answer from the integral representation

$$q_a = q \, \frac{\log\left(\frac{b}{\rho_0}\right)}{\log\left(\frac{a}{b}\right)} = -q \, \frac{\log\left(\frac{b}{\rho_0}\right)}{\log\left(\frac{b}{a}\right)}$$

which is just the simple elegant result from the use of Green's reciprocity theorem.

■ Problem 2.

2) A conducting sphere of radius a and charge Q is brought up toward an infinite grounded conducting plane. As a function of the distance d between the center of the sphere and the plane, the force F attracting the sphere to the plane, is

$$F = \frac{Q^2}{4\pi\varepsilon_0}\frac{1}{4d^2}\left(1 + K_1\frac{a}{d} + K_2\left(\frac{a}{d}\right)^2 + O\left(\left(\frac{a}{d}\right)^3\right)\right)$$

a) (8 points) What is the value of K_1?

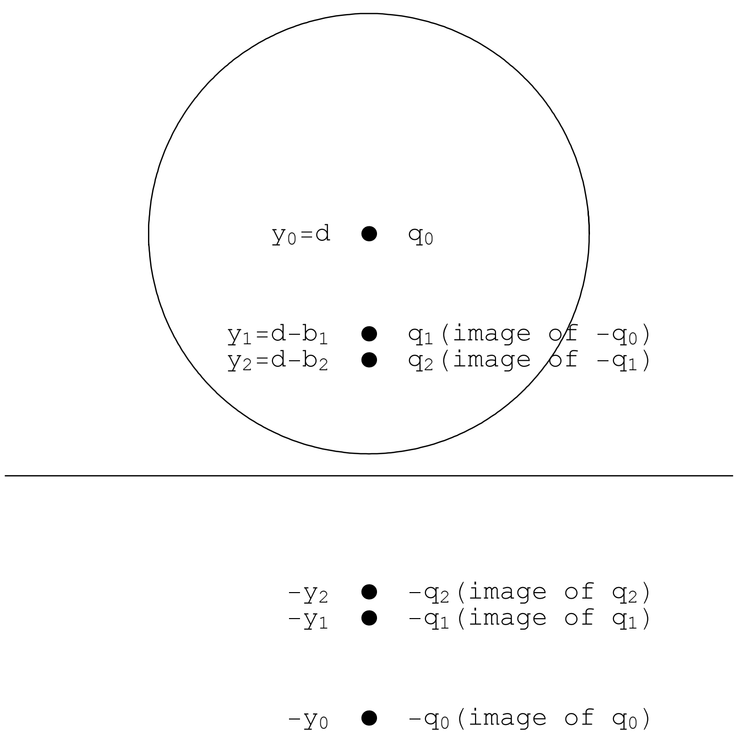
b) (2 points) What is the value of K_2?

■ Solution 2.

Begin graphics

End graphics

In[21]:= **Show[fig2]**



Out[21]= - Graphics -

Use images, starting with a charge q_0 centered in the sphere (so the potential of the sphere is $\frac{1}{4\pi\varepsilon_0}\frac{q_0}{a}$). The figure shows the first few images. The potential of the plane is zero under the influence of the charge pairs

$(q_0, -q_0)$, $(q_1, -q_1)$, $(q_2, -q_2)$, … and similarly the pairs $(-q_0, q_1)$, $(-q_1, q_2)$, $(-q_2, q_{23})$, … give no contribution to the potential of the sphere, so it is at the equipotential produced by q_0.

There is no electric field inside the sphere or below the plane, where all of the equivalent point charges lie. The electric forces act directly in the surface charges on the sphere and plane respectively. If the surface charge density at a point in a surface is σ, then the force on differential area dA acting normal on the surface out of the conductor is $\frac{1}{2\varepsilon_0}\sigma^2 = \frac{1}{2}\varepsilon_0 E_n^2$ where E_n is the normal component of the electric field at the surface. It is geometrically easier to integrate this over the upper surface of the flat plate than over the surface of the sphere. Newton's third law which electrostatics respects means that we can get the total force on the sphere acting toward the grounded plate by calculating the total force on the plate. Thus, if ρ is the distance in the plane from the line along which the images lie, then

$$E_n(\rho) = \frac{2}{4\pi\varepsilon_0}\left(\frac{q_0 y_0}{(\rho^2+y_0^2)^{3/2}} + \frac{q_1 y_1}{(\rho^2+y_1^2)^{3/2}} + \frac{q_2 y_2}{(\rho^2+y_2^2)^{3/2}} + \ldots\right)$$

and so the force acting on the plate directed toward the sphere is (letting $\rho^2 = \xi$)

$$F = \frac{1}{2}\varepsilon_0 \int_0^\infty \frac{4}{16\pi^2\varepsilon_0^2}\left(\frac{q_0 y_0}{(\rho^2+y_0^2)^{3/2}} + \frac{q_1 y_1}{(\rho^2+y_1^2)^{3/2}} + \frac{q_2 y_2}{(\rho^2+y_2^2)^{3/2}} + \ldots\right)^2 2\pi\rho\, d\rho$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int_0^\infty \left(\frac{q_0 y_0}{(\xi+y_0^2)^{3/2}} + \frac{q_1 y_1}{(\xi+y_1^2)^{3/2}} + \frac{q_2 y_2}{(\xi+y_2^2)^{3/2}} + \ldots\right)^2 d\xi.$$

Two types of elementary integrals arise; I'll use *Mathematica* to do the algebra:

In[22]:= $\texttt{Assuming}\left[\{y \in \text{Reals}, y \neq 0\}, \frac{1}{2}\int_0^\infty \frac{q^2 y^2}{(\xi + y^2)^3} d\xi\right]$

Out[22]= $\frac{q^2}{4 y^2}$

In[23]:= $\texttt{Assuming}\left[\{ya \in \text{Reals}, ya > 0, yb \in \text{Reals}, yb > 0\}, \frac{1}{2} 2\int_0^\infty \frac{qa\, ya}{(\xi + ya^2)^{3/2}} \frac{qb\, yb}{(\xi + yb^2)^{3/2}} d\xi\right]$

Out[23]= $\frac{2\, qa\, qb}{(ya + yb)^2}$

Thus, the force drawing the sphere toward the plane is

$$F = F_0 + F_1 + F_2 + \ldots$$

where

$$F_0 = \frac{1}{4\pi\varepsilon_0} q_0\left(\frac{q_0}{(2d)^2} + \frac{q_1}{(2d-b_1)^2} + \frac{q_2}{(2d-b_2)^2} + \ldots\right)$$

$$F_1 = \frac{1}{4\pi\varepsilon_0} q_1\left(\frac{q_0}{(2d-b_1)^2} + \frac{q_1}{(2d-2b_1)^2} + \frac{q_2}{(2d-b_1-b_2)^2} + \ldots\right)$$

$$F_2 = \frac{1}{4\pi\varepsilon_0} q_2\left(\frac{q_0}{(2d-b_2)^2} + \frac{q_1}{(2d-b_1-b_2)^2} + \frac{q_2}{(2d-2b_2)^2} + \ldots\right)$$

which is just the force you would write down if the image charges were real point charges located at the images' places.

Further, by an elementary application of Gauss's law, the total charge of the sphere is

$$Q = q_0 + q_1 + q_2 + \ldots.$$

From the relations for an image in a sphere we have

$$q_1 = q_0 \frac{a}{2d},\ q_2 = q_1 \frac{a}{2d - b_1} = q_0 (\frac{a}{2d})^2 \frac{1}{1-\frac{b_1}{2d}},\ q_3 = q_0\, O(\frac{a}{d})^3,$$

$$b_1 = \frac{a^2}{2d},\ b_2 = \frac{a^2}{2d - b_1} = \frac{a^2}{2d} \frac{1}{1-(\frac{a}{2d})^2},\ \ldots$$

If only K_1 is asked for it is easy to see that

$$F = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_0^2}{(2d)^2} + 2 \frac{q_0^2}{(2d)^2} \frac{a}{2d} + O(a^2) \right) = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{(2d)^2} (1 + \frac{a}{d} + O(a^2))$$

and

$$Q = q_0 (1 + \frac{a}{2d} + O(a^2))$$

so that

$$q_0^2 = Q^2 (1 - \frac{a}{d} + O(a^2)).$$

This gives

$$F = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_0^2}{(2d)^2} + 2 \frac{q_0^2}{(2d)^2} \frac{a}{2d} + O(a^2) \right) = \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{(2d)^2} (1 + \frac{a}{d} + O(a^2))(1 - \frac{a}{d} + O(a^2)) \text{ so that}$$

$$K_1 = 0.$$

To get higher terms, you need to be a bit more systematic. For convenience, set the unit of distance to be d or, in other words, set $d = 1$. At the end the quantities with units are trivially recovered.

Keeping terms only to second order in a, we get

$$F_0 = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} \left(1 + \frac{\frac{a}{2}}{\left(1-\frac{b_1}{2}\right)^2} + \frac{(\frac{a}{2})^2}{\left(1-\frac{b_1}{2}\right)\left(1-\frac{b_2}{2}\right)^2} + O(a)^3 \right) = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} \left(1 + \frac{a}{2} + (\frac{a}{2})^2 + O(a)^3 \right)$$

$$F_1 = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} \frac{a}{2} \left(\frac{1}{\left(1-\frac{b_1}{2}\right)^2} + \frac{a}{2} \frac{1}{(1-b_1)^2} + O(a)^2 \right) = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} \left(\frac{a}{2} + (\frac{a}{2})^2 + O(a)^3 \right)$$

$$F_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} (\frac{a}{2})^2 \frac{1}{1-\frac{b_1}{2}} \left(\frac{1}{\left(1-\frac{b_2}{2}\right)^2} + O(a) \right) = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} (\frac{a}{2})^2 + O(a)^3$$

Adding these you get

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_0^2}{4} \left(1 + a + \frac{3a^2}{4} + O(a)^3 \right).$$

Now we need to express it in terms of the given total charge on the sphere, Q, rather than in terms of the potential that the sphere is set to (which is effectively what the charge q_0 represents – $q_0 = 4\pi\varepsilon_0\, a\, \Phi_{\text{sphere}}$). From above

$$Q = q_0 + q_1 + q_2 + \ldots$$
$$= q_0 + q_0\,\frac{a}{2} + q_0\left(\frac{a}{2}\right)^2 \frac{1}{1-\frac{b_1}{2}} + q_0\,O(a)^3$$
$$= q_0\left(1 + \frac{a}{2} + \left(\frac{a}{2}\right)^2 + O(a)^3\right)$$

so you get

$$q_0^2 = \frac{Q^2}{\left(1+\frac{a}{2}+\left(\frac{a}{2}\right)^2+O(a)^3\right)^2}$$
$$= Q^2\left(1 - 2\left(\frac{a}{2} + \left(\frac{a}{2}\right)^2 + O(a)^3\right) + 3\left(\frac{a}{2} + \left(\frac{a}{2}\right)^2 + O(a)^3\right)^2 + O(a)^3\right)$$
$$= Q^2\left(1 - a - 2\left(\frac{a}{2}\right)^2 + 3\left(\frac{a}{2}\right)^2 + O(a)^3\right)$$
$$= Q^2\left(1 - a + \left(\frac{a}{2}\right)^2 + O(a)^3\right)$$

Finally then,

$$F = \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{4}\left(1 + a + \frac{3a^2}{4} + O(a)^3\right)\left(1 - a + \frac{a^2}{4} + O(a)^3\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{4}\left(1 + a + \frac{3a^2}{4} - a - a^2 + \frac{a^2}{4} + O(a)^3\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{4}\,(1 + O(a)^3)$$

and we get the result that $K_1 = K_2 = 0$.

It is interesting to work out the next term in the series to see if deviation from the zeroth order result happens at this order.

$$Q = q_0 + q_1 + q_2 + q_3 \ldots$$
$$= q_0 + q_0\,\frac{a}{2} + q_0\left(\frac{a}{2}\right)^2 \frac{1}{1-\frac{b_1}{2}} + q_0\left(\frac{a}{2}\right)^3 \frac{1}{1-\frac{b_1}{2}}\,\frac{1}{1-\frac{b_2}{2}} + q_0\,O(a)^4$$
$$= q_0\left(1 + \frac{a}{2} + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^3 + O(a)^4\right),$$

and

$$q_0^2 =$$
$$Q^2\left(1 - 2\left(\frac{a}{2} + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^3 + O(a)^4\right) + 3\left(\frac{a}{2} + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^3 + O(a)^4\right)^2 - 4\left(\frac{a}{2} + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^3 + O(a)^4\right)^3 + O(a)^4\right)$$
$$= Q^2\left(1 - a + \frac{a^2}{4} + O(a)^4\right)$$

Check this arithmetic with *Mathematica* and use it to get the final result.

```
In[24]:= subs =
    {q1 → q0 a/2, q2 → q1 a/(2 - b1), q3 → q2 a/(2 - b2), b1 → a^2/2, b2 → a^2/(2 - b1), b3 → a^2/(2 - b2)};

In[25]:= q02 = 1 / (1 + a/2 + (a/2)^2 1/(1 - b1/2) + (a/2)^3 1/(1 - b1/2) 1/(1 - b2/2) + O[a]^4)^2 //. subs

Out[25]= 1 - a + 0.102941 a^2 + 0.0171748 a^3 + O[a]^4
```

In[26]:=
```
F0 = q0 (q0/4 + (q1/(2 - b1)^2 + q2/(2 - b2)^2 + q3/(2 - b3)^2 //. subs) + O[a]^4)
```

Out[26]= $\frac{q0^2}{4} + 0.209343\, q0^2\, a + 0.15886\, q0^2\, a^2 + 0.0566767\, q0^2\, a^3 + O[a]^4$

In[27]:=
```
F1 = (q1 (q0/(2 - b1)^2 + q1/(2 - 2 b1)^2 + q2/(2 - b1 - b2)^2 + q3/(2 - b1 - b3)^2) //. subs) + O[a]^4
```

Out[27]= $0.209343\, q0^2\, a + 0.210069\, q0^2\, a^2 + 0.171029\, q0^2\, a^3 + O[a]^4$

In[28]:=
```
F2 = (q2 (q0/(2 - b2)^2 + q1/(2 - b2 - b1)^2 + q2/(2 - 2 b2)^2 + q3/(2 - b2 - b3)^2) //. subs) + O[a]^4
```

Out[28]= $0.15886\, q0^2\, a^2 + 0.171029\, q0^2\, a^3 + O[a]^4$

In[29]:=
```
F3 = (q3 (q0/(2 - b3)^2 + q1/(2 - b3 - b1)^2 + q2/(2 - b3 - b2)^2 + q3/(2 - b3 - b3)^2) //. subs) + O[a]^4
```

Out[29]= $0.0566767\, q0^2\, a^3 + O[a]^4$

In[30]:=
```
F = F0 + F1 + F2 + F3
```

Out[30]= $\frac{q0^2}{4} + 0.418685\, q0^2\, a + 0.52779\, q0^2\, a^2 + 0.455412\, q0^2\, a^3 + O[a]^4$

In[31]:=
```
Normal@F /. q0^2 → q02
```

Out[31]= $\frac{1}{4} + 0.168685\, a + 0.13484\, a^2 - 0.0249841\, a^3 + O[a]^4$

Thus we get the first non-trivial term in the series expansion of the force as (putting in the dimensioned things)

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{4d} \left(1 + \frac{1}{2} \left(\frac{a}{d}\right)^3 + O[a]^4\right).$$

Incidentally, we also have an expression for the capacitance (ordinary circuit capacitance) between the sphere and ground. It is

$$C = \frac{Q}{\Phi_{\text{sphere}}} = 4\pi\varepsilon_0\, a\, \frac{Q}{q_0} = 4\pi\varepsilon_0\, a \left(1 + \frac{a}{2d} + \left(\frac{a}{2d}\right)^2 + \left(\frac{a}{2d}\right)^3 + O(a)^4\right).$$

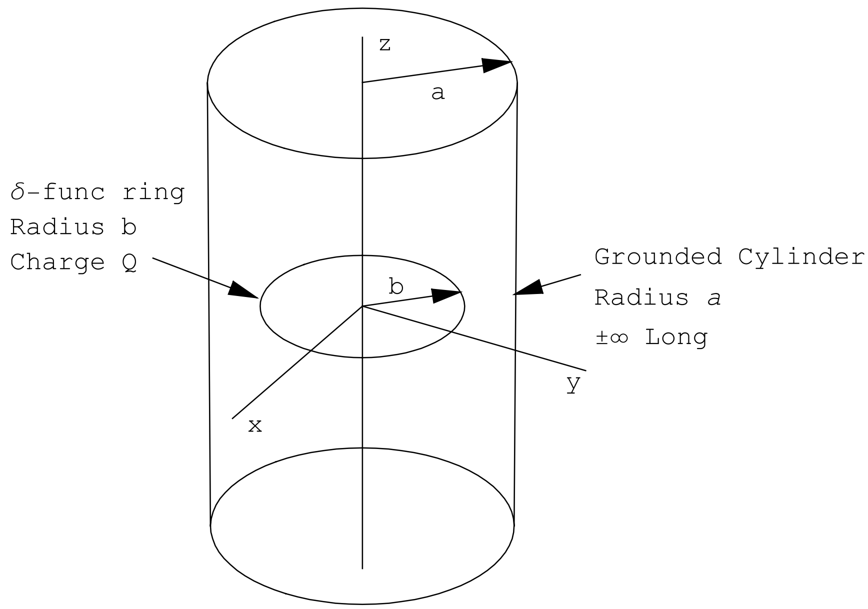
■ Problem 3.

3) (10 points) A conducting hollow infinitely long cylinder of inside radius a is at zero potential and its axis is along the z–axis. A circular ring of line charge of radius $b < a$ has a total charge q uniformly distributed along its perimeter. The charged ring has negligible cross section and so the uniform charge distribution on it may be represented by suitable Dirac delta functions. The ring lies in the $z = 0$ plane and is coaxial with the cylinder. Find the potential everywhere inside the cylinder, expressing the answer in cylindrical coordinates..

Begin graphics

End graphics

```
In[37]:= Show@CylinderWithRing
```



Out[37]= - Graphics -

■ Solution 3.

Solution using non-oscillatory functions in the *z* direction

Since we have azimuthal symmetry and we need a function in ρ with plenty of zeros to match the boundary condition at $\rho = a$, we choose exponentials in z:

$$\Phi(\rho, z) = \sum_{n=1}^{\infty} A_n f_n(z) J_0(k_n \tfrac{\rho}{a})$$

where n counts the zeros of the zero$^{\text{th}}$ order Bessel function with $J_0(k_n) = 0$, and

$$f_n(z) = \begin{pmatrix} e^{-k_n \frac{z}{a}} & z > 0 \\ e^{k_n \frac{z}{a}} & z < 0 \end{pmatrix}.$$

This satisfies Laplace's equation except at $z = 0$ where f(z) has a slope discontinuity: Furthermore, it vanishes as $|z| \to \infty$ as required.

The charge density is $\rho_{\text{charge}} = \frac{q}{2\pi\rho}\,\delta(z)\,\delta(\rho - b)$ which has the proper charge concentration and the proper total charge:

$$\int \rho_{\text{charge}}\, d^3 r = \int_{\rho=0}^{\infty} d\rho \int_{\varphi=0}^{2\pi} \rho\, d\varphi \int_{z=-\infty}^{\infty} dz\, \frac{q}{2\pi\rho}\,\delta(z)\,\delta(\rho - b) = q.$$

To find the values of A_n we must satisfy Poisson's equation:

$$\nabla^2 \Phi = -\frac{\rho_{\text{charge}}}{\varepsilon_0} \;\Rightarrow\; \sum_{n=1}^{\infty} A_n\left(\frac{d^2 f_n}{dz^2} - \left(\frac{k_n}{a}\right)^2 f_n\right) J_0\left(k_n \frac{\rho}{a}\right) = -\frac{q}{2\pi\varepsilon_0 \rho}\, \delta(z)\, \delta(\rho - b).$$

To dissolve the sum, use the orthogonality of the Bessel functions, Jackson formula 3.95:

$$\int_0^a \rho\, J_0\left(k_n \frac{\rho}{a}\right) J_0\left(k_m \frac{\rho}{a}\right) d\rho = \frac{a^2}{2} J_1(k_m)^2\, \delta_{nm}.$$

So multiply both sides of the equation above by $\rho\, J_0(k_m \frac{\rho}{a})$ and integrate over ρ from 0 to a to get

$$A_m\left(\frac{d^2 f_m}{dz^2} - \left(\frac{k_m}{a}\right)^2 f_m\right) \frac{a^2}{2} J_1(k_m)^2 = -\frac{q}{2\pi\varepsilon_0}\, \delta(z)\, J_0\left(k_m \frac{b}{a}\right).$$

This says that f_m has a discontinuity in its derivative at $z = 0$, but is continuous there (were it not continuous, we would have the derivative of a delta function arising from $\frac{d^2 f_m}{dz^2}$). Integrate just over the origin to get

$$A_m \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \left(\left.\frac{d f_m}{dz}\right|_{z=\epsilon} - \left.\frac{d f_m}{dz}\right|_{z=-\epsilon} \right) \frac{a^2}{2} J_1(k_m)^2 = -\frac{q}{2\pi\varepsilon_0} J_0\left(k_m \frac{b}{a}\right),$$

or

$$A_m\left(-\frac{k_m}{a} - \frac{k_m}{a}\right) \frac{a^2}{2} J_1(k_m)^2 = -A_m\, k_m\, a\, J_1(k_m)^2 = -\frac{q}{2\pi\varepsilon_0} J_0\left(k_m \frac{b}{a}\right).$$

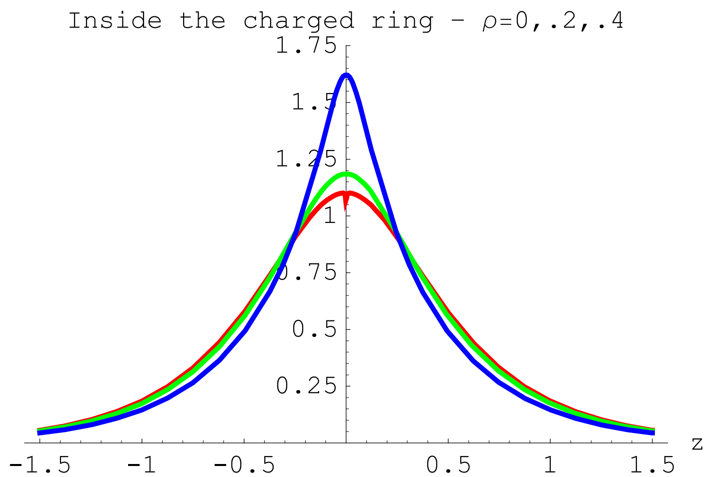
Solving and putting in the potential gives the potential

$$\Phi(\rho, z) = \frac{q}{2\pi\varepsilon_0 a} \sum_{n=1}^{\infty} \frac{1}{k_n}\, e^{-k_n \left|\frac{z}{a}\right|} \frac{J_0\left(k_n \frac{\rho}{a}\right) J_0\left(k_n \frac{b}{a}\right)}{J_1(k_n)^2}.$$

To see its behavior, make some plots, using cgs units (if a is measured in cm and q in cgs charge units, then Φ is in statvolts ... 1statvolt = 300Volts).

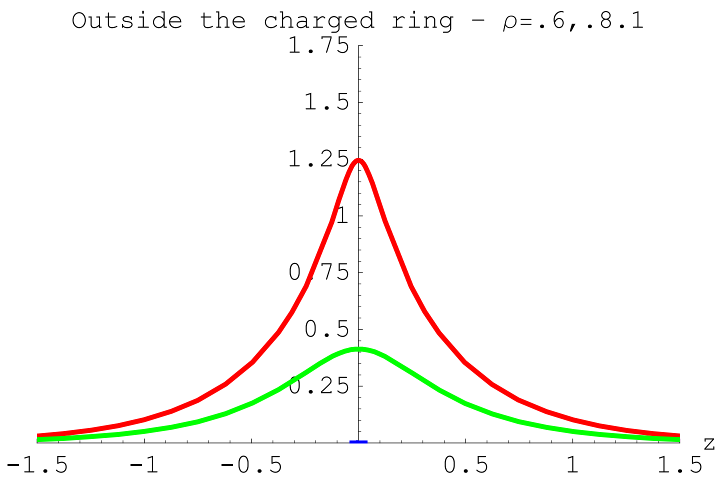
```
In[38]:= << NumericalMath`BesselZeros`

In[39]:= bz = BesselJZeros[0, 100];

In[40]:= Shallow@bz

Out[40]//Shallow=
    {2.40483, 5.52008, 8.65373, 11.7915, 14.9309,
     18.0711, 21.2116, 24.3525, 27.4935, 30.6346, <<90>>}
```

In[41]:=
```
Plot[Evaluate[
  With[{ρ = #, a = 1, q = 1, b = .5}, (2 q)/a Sum[With[{kn = bz[[n]]}, 1/kn e^(-Abs[kn z/a])
        (BesselJ[0, kn ρ / a] BesselJ[0, kn b / a]) / BesselJ[1, kn ]^2], {n, 1, 100}] & /@
    {0, .2, .4}]], {z, -1.5, 1.5}, PlotRange → {0, 1.75},
 PlotStyle → {{RGBColor[1, 0, 0], Thickness[.008]},
   {RGBColor[0, 1, 0], Thickness[.008]}, {RGBColor[0, 0, 1], Thickness[.008]}},
 PlotLabel -> "Inside the charged ring - ρ=0,.2,.4",
 AxesLabel → {"z", ""}]
```



Out[41]= - Graphics -

In[42]:=
```
Plot[Evaluate[
  With[{ρ = #, a = 1, q = 1, b = .5}, (2 q)/a Sum[With[{kn = bz[[n]]}, 1/kn e^(-Abs[kn z/a])
      (BesselJ[0, kn ρ / a] BesselJ[0, kn b / a]) / BesselJ[1, kn ]^2], {n, 1, 100}]] & /@
    {.6, .8, 1}]], {z, -1.5, 1.5}, PlotRange → {{-1.5, 1.5}, {0, 1.75}},
 PlotStyle → {{RGBColor[1, 0, 0], Thickness[.008]},
   {RGBColor[0, 1, 0], Thickness[.008]}, {RGBColor[0, 0, 1], Thickness[.008]}},
 PlotLabel -> "Outside the charged ring - ρ=.6,.8.1",
 AxesLabel → {"z", ""}]
```



Out[42]= - Graphics -

Solution using non-oscillatory functions in the ρ direction

I did not expect people to choose this solution, but in this case, we get a solution in terms of an integral. The form is

$$\Phi = \int_0^\infty A(k) \cos kz\, g(k\rho)\, dk$$

where

$$g(k\rho) = \begin{pmatrix} I_0(k\rho)\,(I_0(kb)\,K_0(ka) - I_0(ka)\,K_0(kb)) & 0 \le \rho < b \\ I_0(kb)\,(I_0(k\rho)\,K_0(ka) - I_0(ka)\,K_0(k\rho)) & b < \rho \le a \end{pmatrix}.$$

This is constructed from non-oscillatory Bessel functions to be regular at $\rho = 0$ (so only the I_0 function inside the hoop), to be continuous at $\rho = b$, and to vanish at $\rho = a$. Of course, since there is no dependence on φ only the zeroth order functions are involved. Now we need

$$\nabla^2 \Phi = \int_0^\infty A(k') \cos k' z \left(\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg(k'\rho)}{d\rho}\right) - k'^2\, g(k'\rho)\right) dk' = -\frac{q}{2\pi\varepsilon_0\,\rho}\, \delta(z)\, \delta(\rho - b)$$

Multiply both sides by $\cos kz$ and integrate over all z, using

$$\int_{-\infty}^{\infty}\cos k\,z\cos k'\,z\,dz = \frac{1}{4}\,2\,\pi\,(\,\delta(k+k')+\delta(k-k')+\delta(-k+k')+\delta(-k-k')\,)$$
$$=\pi\,(\delta(k+k')+\delta(k-k')) = \begin{pmatrix}\pi\,\delta(k-k') & k\neq 0\\ 2\,\pi\,\delta(k') & k=0\end{pmatrix} = t(k)\,\pi\,\delta(k-k') \text{ where } t(k)=\begin{pmatrix}1 & k>0\\ 2 & k=0\end{pmatrix}$$

to get

$$\int_0^{\infty}A(k')\,t(k)\,\pi\,\delta(k-k')\left(\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{dg(k'\rho)}{d\rho}\right)-k'^2\,g(k'\rho)\right)dk'$$
$$=A(k)\,t(k)\,\pi\,\left(\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{dg(k\rho)}{d\rho}\right)-k^2\,g(k\rho)\right)$$
$$=-\frac{q}{2\,\pi\,\varepsilon_0\,\rho}\,\delta(\rho-b)$$

Integrating over ρ from $b-\epsilon$ to $b+\epsilon$, $\epsilon>0$, and taking the limit as $\epsilon\to 0$ gives

$$A(k) = -\frac{q}{2\,\pi\,\varepsilon_0}\;\frac{1}{t(k)\,\pi\,k\,b\,G(k)}$$

where

$$G(k)=I_0(kb)\,(I_0'(kb)\,K_0(ka)-I_0(ka)\,K_0'(kb))-I_0'(kb)\,(I_0(kb)\,K_0(ka)-I_0(ka)\,K_0(kb)).$$

As usual, we use the fact that both $I_0(k\rho)$ and $I_0(k\rho)\,K_0(ka)-I_0(ka)\,K_0(k\rho)$ satisfy the same Stürm–Liouville equation (with $p(x)=x$) so $G(k)=\frac{\text{cnst}}{k\,b}$. Evaluate the constant for the functions used in *Mathematica:*

```
In[57]:= FullSimplify[BesselI[0, x]
            ((∂x BesselI[0, x]) BesselK[0, k a] - BesselI[0, k a] ∂x BesselK[0, x])
          - (∂x BesselI[0, x]) ((BesselI[0, x]) BesselK[0, k a] -
            BesselI[0, k a] BesselK[0, x])]

Out[57]= BesselI[0, a k]
         ----------------
                x
```

So we get the simpler expression

$$G(k)=\frac{I_0(ka)}{k\,b}.$$

The result is

$$\Phi=-\frac{q}{2\,\pi^2\,\varepsilon_0}\int_0^{\infty}\frac{\cos kz\,g(k\rho)}{I_0(ka)}\,dk$$

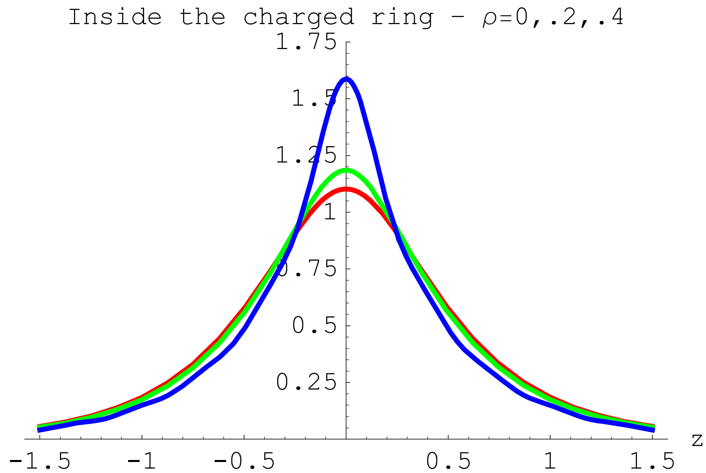
where

$$g(k\rho)=\begin{pmatrix}I_0(k\rho)\,(I_0(kb)\,K_0(ka)-I_0(ka)\,K_0(kb)) & 0\le\rho<b\\ I_0(kb)\,(I_0(k\rho)\,K_0(ka)-I_0(ka)\,K_0(k\rho)) & b<\rho\le a\end{pmatrix}.$$

I have dropped the $t(k)$ which is relevant only when there is a concentration at $k=0$.

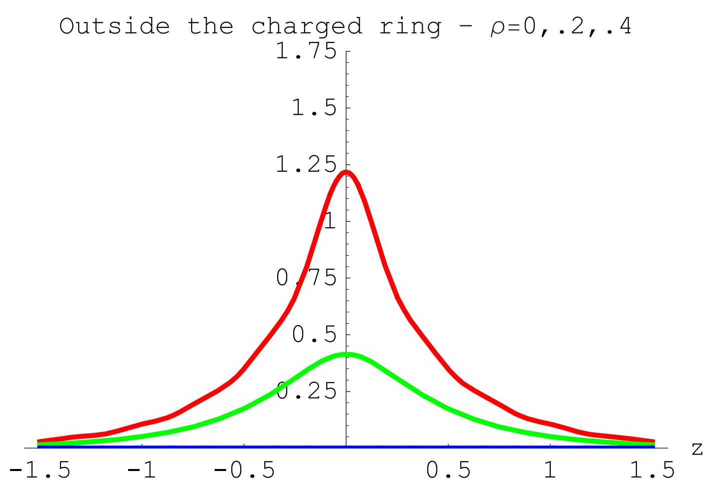
```
In[43]:= g[ρ_, k_, a_, b_] := If[ρ < b, BesselI[0, k ρ]
            (BesselI[0, k b] BesselK[0, k a] - BesselI[0, k a] BesselK[0, k b]), BesselI[
             0, k b] (BesselI[0, k ρ] BesselK[0, k a] - BesselI[0, k a] BesselK[0, k ρ])]
```

In[58]:=
```
Plot[Evaluate[With[{ρ = #, a = 1, q = 1, b = .5},
    -(2 q/π) NIntegrate[(Cos[k z] g[ρ, k, a, b])/(BesselI[0, k a]), {k, 0, 20}] & /@ {0, .2, .4}]],
  {z, -1.5, 1.5}, PlotRange → {0, 1.75},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[.008]},
    {RGBColor[0, 1, 0], Thickness[.008]}, {RGBColor[0, 0, 1], Thickness[.008]}},
  PlotLabel -> "Inside the charged ring - ρ=0,.2,.4", AxesLabel → {"z", ""}]
```



Out[58]= - Graphics -

In[60]:=
```
Plot[Evaluate[With[{ρ = #, a = 1, q = 1, b = .5},
    -(2 q/π) NIntegrate[(Cos[k z] g[ρ, k, a, b])/(BesselI[0, k a]), {k, 0, 20}] & /@ {.6, .8, 1}]],
  {z, -1.5, 1.5}, PlotRange → {0, 1.75},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[.008]},
    {RGBColor[0, 1, 0], Thickness[.008]}, {RGBColor[0, 0, 1], Thickness[.008]}},
  PlotLabel -> "Outside the charged ring - ρ=0,.2,.4", AxesLabel → {"z", ""}]
```



Out[60]= - Graphics -

More work could be done to improve the numerical evaluations of the integral (*Mathematica* has various complaints in its evaluation), but from the above it is quite reasonable that the integral representation of the potential is the same as the series.

A Note on Bessel Functions

Several people carried over to Bessel functions a relation that is correct for sinusoids, but not for Bessel functions.

It is familiar with sinusoids that if $S(kx)$ satisfies the sinusoid equation, $\partial_{x,x} S + k^2 S = 0$, then so does $S(k(x-a))$ where a is any constant. In boundary value problems this is often useful when the potential is zero on a boundary at a in a variable along which the solution is sinusoidal. Then instead of writing $(A \sin kx + B \cos kx)$ and solving for A and B to get a zero at $x = a$ $(A = \alpha \cos ka,\ B = -\alpha \sin ka)$ you just write down immediately, $\alpha \sin(k(x-a))$.

This make it tempting to do a similar thing with Bessel functions. Namely, if $m \neq 0$ and you need a zero at $\rho = a$ in a cylindrical BC problem for which the Bessel functions $J_m(k\rho)$ are appropriate, then some people used $J_m(k(\rho - a))$ as a solution. Unfortunately, this function, while zero at $\rho = a$, is not a solution of Bessel's equation. Thus with Bessel functions, you need, in general, to explicitly use both linearly independent solutions of the 2nd order differential equation, say $B_m(k\rho)$ and $C_m(k\rho)$ to construct a soution which vanishes at $\rho = a$. It is very easy to arrange this:

$$\alpha(B_m(k\rho)\, C_m(ka) - B_m(ka)\, C_m(k\rho))$$

does the job. But this is NOT $B_m(k(\rho - a))$, for example.

In the sinusoid case, the differential equation has the symmetry that it is unchanged under the simple transformation $x \to y = x - a$ for constant a. The reason is simply that $\frac{d^2}{dx^2} = \frac{d^2}{dy^2}$ and so the equation in y, $\frac{d^2 S(ky)}{dy^2} + k^2 S(ky) = 0$ is just the same as that in x. So if $S(kx)$ satisfies the equation in x, then $S(ky) = S(k(x-a))$ does also.

So let's try it with Bessel's equation, which is

$$\frac{1}{\rho}\, \partial_\rho\, (\rho\, \partial_\rho\, B_m(k\rho)) - \frac{m^2}{\rho^2}\, B_m(k\rho) + k^2\, B_m(k\rho) = 0,$$

or

$$k^2\, B''_m(k\rho) + \frac{k}{\rho}\, B'_m(k\rho) - \frac{m^2}{\rho^2}\, B_m(k\rho) + k^2\, B_m(k\rho) = 0,$$

or

$\frac{d^2}{dx^2} B_m(x) + \frac{1}{x} \frac{d}{dx} B_m(x) - \frac{m^2}{x^2} B_m(x) + B_m(x) = 0.$

Under the substitution $x \to y = x - a$ we get $\frac{d}{dx} = \frac{d}{dy}$ and $\frac{d^2}{dx^2} = \frac{d^2}{dy^2}$ but the differential equation becomes

$\frac{d^2}{dy^2} (B_m(y+a)) + \frac{1}{y+a} \frac{d}{dy} (B_m(y+a)) - \frac{m^2}{(y+a)^2} B_m(y+a) + B_m(y+a) = 0$

which is simply another differential equation, **not Bessel's**. So we see that Bessel functions don't share with sinusoids a simple, so-called addition formula.

There do exist addition formulas for Bessel functions but we don't need them in this course. See Abramowitz and Stegun formulas 9.1.75 and 9.1.80 or Gradshteyn and Ryzhik section 8.53.