■ Green function for the half space defined by a plane use a non-discrete set of separation variables

Of course we already know the Green function for this problem by using images. This then is an illustration of how to use a non-discrete set of separation constants.

Formulate the problem in cartesian coordinates. Then Laplace's equation gives us real and imaginary exponentials. If the plane is $z = 0$, then it is convenient to choose the functions as $e^{i k_x x}$, $e^{i k_y y}$, $e^{\pm k z}$ where $k = \sqrt{k_x^2 + k_y^2}$. It is important to always remember that k is not a "free" separation constant if we choose k_x and k_y to be free - Laplace's equation requires $-k_x^2 - k_y^2 + k^2 = 0$. there are no finite boundaries in x or y that require that k_x and k_y to be certain discrete sets, so we take as our trial function

$$G(x, y, z; x', y', z') = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x\, A(k_x, k_y)\, g(z, k)\, e^{i k_x (x-x')}\ e^{i k_y (y-y')}\ .$$

The introduction of the factors $e^{-i k_x x'}$ and $e^{-i k_x y'}$ ismotivated by the know symmetry of a green function under the interchange of $\vec{r}$ and $\vec{r}'$ (and the fact that in the end only the real part of the complex functions given will be relevant). Now we must construct $g(z, k)$ so that it is continuous at $z = z'$, vanishes at $z = 0$ and at $z = \infty$, and just uses $e^{\pm k z}$. The appropriate function is easily seen to be

$$g = \left\{ \begin{array}{ll} e^{-k z'} \sinh k z, & \text{for } 0 \le z \le z' \\ e^{-k z} \sinh k z', & \text{for } z' \le z \end{array} \right\}, \qquad \text{eq.1}$$

To make the formulas more concise, it is convenient to use the notation

$$\vec{k} = k_x \vec{e}_x + k_y \vec{e}_y,\ \ \vec{\rho} = x \vec{e}_x + y \vec{e}_y,\ \ \vec{\rho}' = x' \vec{e}_x + y' \vec{e}_y.$$

Then the z separation constant, k, is just the length of the vector $\vec{k}$. Furthermore, $dk_x\, dk_y = d^2 k$. Then we have

$$G = \int A(\vec{k})\, g\, e^{i \vec{k} \cdot (\vec{\rho} - \vec{\rho}')}\, d^2 k$$

where the integral is over the whole plane.

Now we use Poisson's equation to write

$$\nabla^2 G = \int A(\vec{k}) \left(-k^2 g + \frac{d^2 g}{d z^2}\right) e^{i \vec{k} \cdot (\vec{\rho} - \vec{\rho}')}\, d^2 k = -\frac{1}{\varepsilon_0}\, \delta(z - z')\, \delta^2(\vec{\rho} - \vec{\rho}'), \qquad \text{eq. 2}$$

To dissolve the integral in eq. 1 into a single term, we need the integral form of the completeness equation for sinusoids. So a detour to derive it.

■ Integral completeness theorem for sinusoids

For a discrete set of separation constants, we have the completeness relation

$\delta(x) = \frac{1}{2d} \sum_{n=-\infty}^{\infty} e^{i n \pi x/d}$ with $-d \leq x \leq d$.

Now I want to take d to infinity. So writing $k_n = n \frac{\pi}{d}$ and $\Delta k = \frac{\pi}{d}$ (associated with $\Delta n = 1$ in the sum), we get

$\delta(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i k_n x} \Delta k$.

As we let $d \to \infty$, $\Delta k \to dk$ and the sum turns into an integral. Thus we get in the limit

$\int_{-\infty}^{\infty} e^{i k x} dk = 2\pi \delta(x)$, and changing symbols, $\int_{-\infty}^{\infty} e^{i k x} dx = 2\pi \delta(k)$.

Since the integral is real, its complex conjugate is itself, so we also have
$\int_{-\infty}^{\infty} e^{-i k x} dk = 2\pi \delta(x)$, and $\int_{-\infty}^{\infty} e^{-i k x} dx = 2\pi \delta(k)$.

■ Back to the Green function

To dissolve the integral in eq. 2, multiply both sides of the equation by $e^{-i \vec{k}' \cdot(\vec{\rho}-\vec{\rho}')}$ to get

$$\int A(\vec{k}) \left(-k^2 g + \frac{d^2 g}{d z^2}\right) e^{i (\vec{k}-\vec{k}') \cdot (\vec{\rho}-\vec{\rho}')} d^2 k = -\frac{1}{\varepsilon_0} \delta(z-z') e^{-i \vec{k}' \cdot (\vec{\rho}-\vec{\rho}')} \delta^2(\vec{\rho}-\vec{\rho}')$$

and then integrate over all of $\vec{\rho}$. Using the completeness r3lation for sinusoids, we get
$(2\pi)^2 A(\vec{k}') \left(-k'^2 g + \frac{d^2 g}{d z^2}\right) = -\frac{1}{\varepsilon_0} \delta(z-z')$.

Now integrate over z' in z and use continuity of g to get

$$(2\pi)^2 A(\vec{k}') \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \left(\frac{dg}{dz} \Big|_{z=z'+\epsilon} - \frac{dg}{dz} \Big|_{z=z'-\epsilon} \right) = -\frac{1}{\varepsilon_0}.$$

Using eq. 1 we get

$$(2\pi)^2 A(\vec{k}') (-k' e^{-k' z'} \sinh k' z' - k' e^{-k' z'} \cosh k' z') = -k' (2\pi)^2 A(\vec{k}') = -\frac{1}{\varepsilon_0},$$

or $A(\vec{k}) = \frac{1}{(2\pi)^2 \varepsilon_0} \frac{1}{k}$.

Finally with $z_>$ taken for the greater of z and z' and $z_<$ the lesser, the Green function is

$$G(\vec{\rho}, z; \vec{\rho}', z') = \frac{1}{(2\pi)^2 \varepsilon_0} \int \frac{1}{k} e^{-k z_>} \sinh(k z_<) e^{i \vec{k} \cdot (\vec{\rho}-\vec{\rho}')} d^2 k.$$

It is convenient to get rid of the singularity at $k = 0$ by going po polar coordinates in $\vec{k}$ space, i.e., let $k_x = k \cos\varphi_k$, $k_y = k \sin\varphi_k$ so $d^2 k = k\, dk\, d\varphi_k$. After a little fiddling you get the nicer, manifestly real expression

$$G(\vec{\rho}, z; \vec{\rho}', z') = \frac{1}{\pi^2 \varepsilon_0} \int_0^{2\pi} d\varphi_k \int_0^{\infty} dk\, e^{-k z_>} \sinh(k z_<) \cos(k |\vec{\rho}-\vec{\rho}'| \cos\varphi_k).$$

It is not at all obvious, but it is true that this is an obscure way to write

$$\frac{1}{4\pi\varepsilon_0} \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \right).$$

Here are some numerical evaluation that demonstate this (in special cases. of course).

```
point = {x → .5, y → 1., xp → 0, yp → 0, zp → 1};
zvalues = {.5, 1.5, 2., 2.5, 3.};
```

```
With[{z = #}, With[{kmax = 10.}, NIntegrate[
    1/π^2 (Evaluate[If[z > zp, ⅇ^(-k z) Sinh[k zp], ⅇ^(-k zp) Sinh[k z]]] Cos[k Sqrt[x^2 + y^2] Cos[ϕ]]) /.
     point, {k, 0, kmax}, {ϕ, 0, π / 2}]]] & /@ zvalues
```

```
{0.0223892, 0.0358676, 0.0281957, 0.0208777, 0.0155705}
```

$$\text{With}\left[\{z = \#\},\ \frac{1}{4\pi}\left(\frac{1}{\sqrt{x^2 + y^2 + (z - zp)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + zp)^2}}\right) /.\ \text{point}\right]\ \&\ /@\ \text{zvalues}$$

```
{0.0224388, 0.0359171, 0.0281958, 0.0208777, 0.0155705}
```

The comparison is not at all sophisticated is choice of the range of the k integration, and so at $z = .5$ and 1.5 it doesn't do a very good job, but at the other points, it looks OK.