Ph196bEM Homework 1 Solutions - 2004

1. Jackson (3rd Edition), chapter 1, problem 1.1.

Use Green's theorem [and (1,21) if necessary] to prove the following:

a) Any excess charge placed on a conductor must be entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

b) A closed hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from fields due to charges placed inside it.

c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ / ε_0, where σ is the charge density per unit area on the surface.

■ Solution Jackson Problem 1.1

a) If there were charge inside a conductor, then by Gauss' law there would have to be an electric field inside it also. But this would produce a current, i.e., moving charges, and so the problem would not be electrostatics. This argument does not apply to the surface of a conductor, where charges can indeed collect. Physically, there must be a finite depth within which the idealized classical "surface charge" resides; its calculation is intrinsically quantum mechanical since the atomic structure of the conductor must be taken into account. The depth is of order the size of atoms, i.e. Angstroms.

b) For a closed hollow conductor that has no charges in its hollow, the potential inside the hollow must be a constant by the uniqueness theorem of electrostatics. Note that we have a case in which a volume is defined by an equipotential and so one solution is that the whole of the volume is at that potential. Since this is A solution, it is THE solution by the uniqueness theorem. Thus the volume inside a closed hollow conductor is shielded from the electric field of any charge outside the conductor. If there is charge inside the volume, there must be an electric field in the hollow by Gauss' law. Further, application of Gauss' law to any surface completely inside the metal and enclosing the hollow yields zero since $E = 0$ inside the conductor. Thus any charge in the hollow induces an equal and opposite charge on the inside surface. The field inside the hollow terminates on this inside surface, creating a surface charge density on it.

c) There can be no tangential component of electric field on the surface of a conductor since, by Stoke's Theorem applied to $\vec{\nabla} \times \vec{E} = 0$, that would imply a tangential field just inside the surface and a concomitant tangential current. This does not constrain the normal component of the field, however. Apply $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$ to a small volume with negligible thickness normal to the surface and relatively large flats with area $\overrightarrow{dA}$ in the direction of the outward normal to the surface on the outside and , $-\overrightarrow{dA}$ on the inside. Then, use the fact that there is no field inside the conductor to get $\sigma = \varepsilon_0 E_n$ as the surface charge density where E_n is the component of $\vec{E}$ in the direction of the outward normal to the surface.

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2. Lines of Force

An electric field is often represented by a map of its lines of force.

A line of force is defined by a curve $\vec{r}(s)$ satisfying, at each point where $E = |\vec{E}|$ is finite and non-zero, $\frac{d r_i}{d s} = \frac{E_i}{E}$. A point were $E = 0$ is called a neutral point and there is no unique line of force through it; similarly at a point charge q there is not a unique line of force. You can think of lines of force beginning or ending at these special points.

Further a tube of flux in a charge-free region is defined by an open surface $\mathcal{S}$ with perimeter $\mathcal{P}$ containing no neutral points or point charges and the set of all lines of force through the points of $\mathcal{P}$. The flux in the tube is defined by $\mathcal{F} = \int_{\mathcal{S}} \vec{E} \cdot d\vec{A}$ with $d\vec{A}$ defined over $\mathcal{S}$ with one sense or the other; it is easy to see that the flux through any cross section $\mathcal{S}'$ of the tube is just $\mathcal{F}$ as no flux passes into or out of the tube through the wall defined by the lines of force.

In geometrically symmetric situations, tubes of flux are often easy to calculate and may be used to simplify the calculation of lines of force.

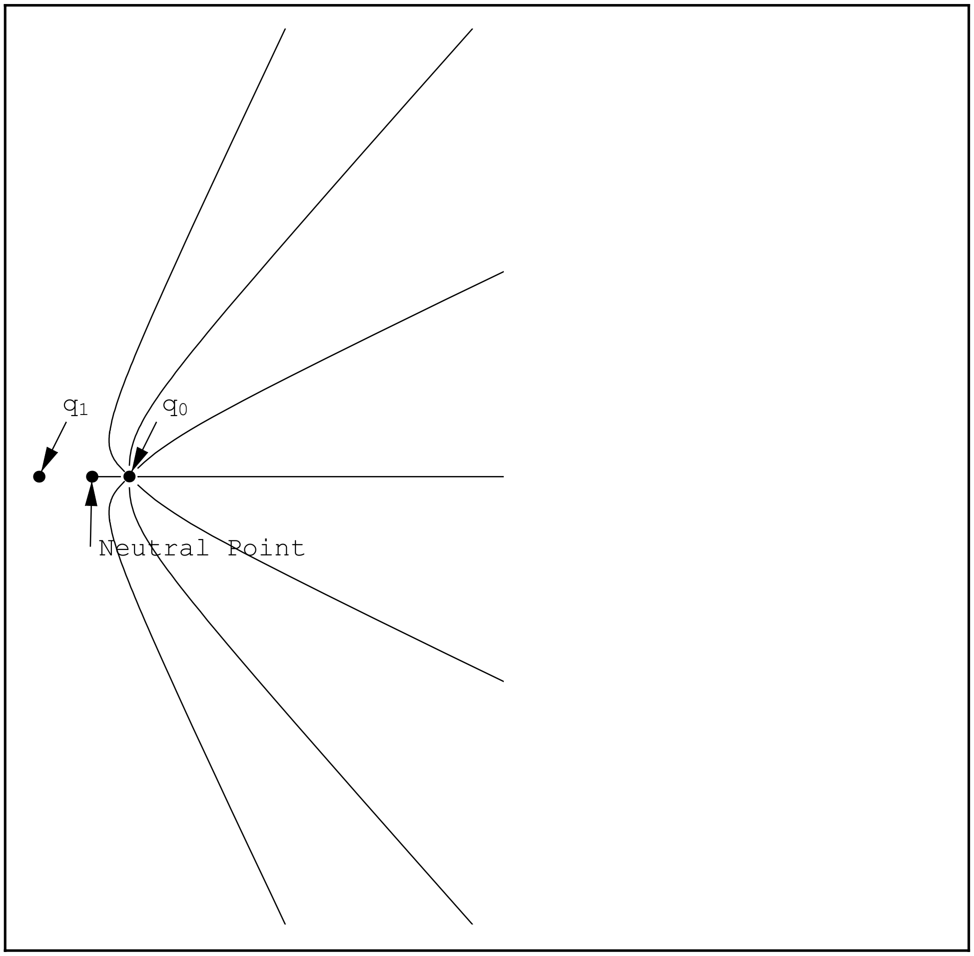
As a simple example, a cone of half-angle θ with vertex on a point charge q carries a flux of $\frac{q}{2\varepsilon_0}(1 - \cos\theta)$ and the lines of force are radial from the charge. You can think of the surface $\mathcal{S}$ as a spherical cap over the end of a truncated cone centered at the origin having a radius ϵ of arbitrarily small value.

Suppose $q_0 = 1$ is located at the origin and $q_1 = 2$ is located at $r_1 = (0, 0, -1)$. A line of force makes an angle θ with the $+z$-axis as it approaches the origin. Find the angle ψ that this line makes with the $+z$-axis as it recedes to infinity.

■ Solution - Lines of Force

A tube of flux starting on a point charge $q_o = 1$ with angle θ to the $+z$-axis at the origin, as indicated in the (carefully calculated) sketch, carries a flux of $\frac{1}{2\varepsilon_0}(1 - \cos\theta)$. (Strictly speaking, the tube of flux is defined by the flux through a spherical cap of infinitesimal radius from the origin capping the truncated cone of half angle θ.)

Lines of Force for $q_0 = 1$ at origin and $q_1 = 2$ at {0,0,-1}. Only lines starting on q_0 are shown.



By simple superposition, the flux through a disk whose normal is the z-axis and whose radius is ρ is just

$$\frac{1}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+\rho^2}}\right)+\frac{2}{2\varepsilon_0}\left(1-\frac{z+1}{\sqrt{(z+1)^2+\rho^2}}\right)$$

and the point (z, ρ) will be on the tube of flux if this is equal to $\frac{1}{2\varepsilon_0}(1-\cos\theta)$. As the line of force approaches infinity we have both

$\frac{z}{\sqrt{z^2+\rho^2}} \to \cos\psi$ and $\frac{z+1}{\sqrt{(z+1)^2+\rho^2}} \to \cos\psi$.

Therefore

$(1-\cos\theta) = 3\,(1-\cos\psi) = 2\sin^2\frac{\theta}{2} = 6\sin^2\frac{\psi}{2}$,

or

$\psi = 2\sin^{-1}\left(\frac{1}{\sqrt{3}}\sin^2\frac{\theta}{2}\right)$.

Although it was not asked for, it is easy to see from the "sketch" (and to show analytically) that the lines of force point back to the neutral point on the z axis at $z = -\sqrt{2} + 1$.

3. Jackson, Ch. 1 problem 1.4.

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge Q. Use Gauss' s theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -1, +2$.

■ Solution - Jackson problem 1.4.

Since any rotation about the origin in any of the three cases given leaves the physical situation unchanged, the electric field must have only a radial component. For any fixed radius $r \neq a$, we have from Maxwell's equation,

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$,

Gauss's law

$\int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \int_{\partial V} \vec{E}\cdot d\vec{A}$,

and symmetry that

$E(r) = \frac{Q(r)}{4\pi\varepsilon_0 r^2}$,

where $E(r)$ is the radial component of $\vec{E}$ at radius r and $Q(r)$ is the amount of charge inside the radius r. This relation may also be used at surface $r = a$ if the electric field is continuous there. It is not continuous in the conducting sphere case there is a delta function of charge on the surface), but it is in the other two (no delta function of charge on the $r = a$ surface).

Conducting sphere case

From the above, we get $E(r) = 0$ for $r < a$ since there is no charge inside the sphere (all charge is on its outside surface) and

$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$ for $r > a$.

It is usual to define the electric field at $r = a$ as the limit

$$E(a) = \lim_{r\to 0,\, r>0} E(r) = \frac{Q}{4\pi\varepsilon_0 a^2}$$

Uniform Charge Distribution

The charge density ρ is zero for $r > a$ and

$$\frac{3Q}{4\pi a^3}$$

for $r < a$. Thus $Q(r) = \frac{3Q}{4\pi a^3}\,\frac{4\pi}{3}\,r^3 = Q\left(\frac{r}{a}\right)^3$ for $r < a$. Thus the radial electric field is

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r}{a}\right)^3 = \frac{Q}{4\pi\varepsilon_0 a^2}\,\frac{r}{a} \text{ for } r < a \text{ and } \frac{Q}{4\pi\varepsilon_0 r^2} \text{ for } r > a.$$

Power Law Charge Density

If $\rho = C\left(\frac{r}{a}\right)^n$ for $n > -3$, and the total charge inside radius a is Q,
then

$$Q = \int_0^a C\left(\frac{r}{a}\right)^n 4\pi r^2\, dr = 4\pi C a^{-n} \int_0^a r^{n+2}\, dr = \frac{4\pi C}{n+3}\, a^{-n}\, a^{n+3}\,.$$

Note that the logarithmic possibility if n were -3 is avoided by the $n > -3$ condition. Solve this for C to get

$\rho = \frac{(n+3)\,Q}{4\,\pi\,a^3}\,(\frac{r}{a})^n.$

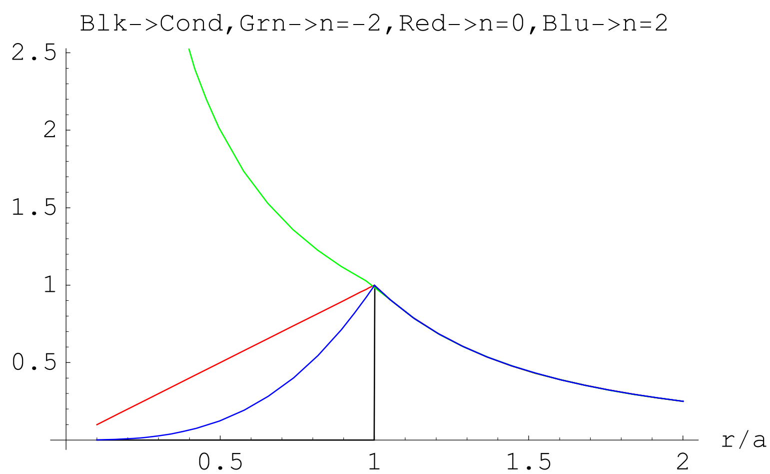
So

$Q(r) = \int_0^r \frac{(n+3)\,Q}{4\,\pi\,a^3}\,(\frac{r}{a})^n\,4\,\pi\,r^2\,dr = Q\,(\frac{r}{a})^{n+3}$

and

$E(r) = \frac{Q}{4\,\pi\,\varepsilon_0\,r^2}\,(\frac{r}{a})^{n+3} = \frac{Q}{4\,\pi\,\varepsilon_0\,a^2}\,(\frac{r}{a})^{n+1}$ for $r < a$ and $\frac{Q}{4\,\pi\,\varepsilon_0\,r^2}$ for $r > a$.

```
In[1]:= Plot[{UnitStep[r - 1]/r^2, If[r < 1, r, 1/r^2], If[r < 1, r^-1, 1/r^2], If[r < 1, r^3, 1/r^2]},
   {r, .1, 2}, PlotStyle → {GrayLevel[0], RGBColor[1, 0, 0],
     RGBColor[0, 1, 0], RGBColor[0, 0, 1]}, AxesLabel → {"r/a", None},
   PlotLabel -> "Blk->Cond,Grn->n=-2,Red->n=0,Blu->n=2"]
```



Out[1]= - Graphics -

4. Jackson, Ch. 1 problem 1.12.

Prove *Green's reciprocation theorem:* If Φ is the potential due to a volume-charge density ρ within a volume V and the surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\,\Phi'\,d^3r + \int_S \sigma\,\Phi'\,dA = \int_V \rho'\,\Phi\,d^3r + \int_S \sigma'\,\Phi\,dA$$

■ Solution - Green's Reciprocation Theorem, Jackson Problem 1.12

Let Φ be the potential in a volume V produced by a space charge density ρ in V and a surface charge density σ on the *conducting* surface ∂V of V. Then we know that inside V

$\nabla^2 \Phi = -\rho / \varepsilon_0$.

In exactly the same geometry, let Φ' be produced by ρ' and σ' so that inside V

$\nabla^2 \Phi' = -\rho' / \varepsilon_0$.

Then, "cross multiplying" and subtracting these equations yields

$$\Phi' \nabla^2 \Phi - \Phi \nabla^2 \Phi' = -\frac{1}{\varepsilon_0} (\Phi' \rho - \Phi \rho').$$

But,

$$\vec{\nabla} \cdot \left(\Phi' \vec{\nabla} \Phi\right) = \vec{\nabla} \Phi' \cdot \vec{\nabla} \Phi + \Phi' \nabla^2 \Phi$$

and similarly

$$\vec{\nabla} \cdot \left(\Phi \vec{\nabla} \Phi'\right) = \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi' + \Phi \nabla^2 \Phi'.$$

Subtracting, the symmetric term disappears and so we get

$$\Phi' \nabla^2 \Phi - \Phi \nabla^2 \Phi' = \vec{\nabla} \cdot \left(\Phi' \vec{\nabla} \Phi - \Phi \vec{\nabla} \Phi'\right) = -\frac{1}{\varepsilon_0} (\Phi' \rho - \Phi \rho').$$

Next integrate the divergence term and the last term over just the inside of V to get

$$\int_{\partial V} \left(\Phi' \vec{\nabla} \Phi - \Phi \vec{\nabla} \Phi'\right) \cdot d\vec{A} = -\frac{1}{\varepsilon_0} \int_V (\Phi' \rho - \Phi \rho') \, d^3 x \qquad \text{(eq 1).}$$

Then, use the fact that the surface charge density on the conductor satisfies

$$\frac{\sigma}{\varepsilon_0} = \vec{n} \cdot \vec{E} = -\vec{n} \cdot \vec{\nabla} \Phi$$

where $\vec{n}$ is a unit vector pointing *away* from the conductor and so *into* the volume V. Thus for the case at hand, $-\vec{n}$ is a unit vector in the direction of the $d\vec{A}$ in eq. 1. We finally get,

$$\frac{1}{\varepsilon_0} \int_{\partial V} (\Phi' \sigma - \Phi \sigma') \, dA = -\frac{1}{\varepsilon_0} \int_V (\Phi' \rho - \Phi \rho') \, d^3 x$$

or

$$\int_V \Phi' \rho \, d^3 x + \int_{\partial V} \Phi' \sigma \, dA = \int_V \Phi \rho' \, d^3 x + \int_{\partial V} \Phi \sigma' \, dA \, .$$

Notice that the surface ∂V may consist of several different conductors, each at a different potential. This and similar results have interesting and useful application to different electrical situations in a particular geometry.

An example is that the geometry is a pair of coaxial cylinders od radii a and b, $b > a$. In the unprimed case, the outer cylinder is grounded, the inner is at potential V, and $\rho = 0$. In the primed case, the potentials are the same, but the charge density is a unit point charge at radius c, $a < c < b$. This is the situation in a cylindrical ionization chamber.

$$V \int_{\text{inner cylinder}} \sigma \, dA = \Phi(c) + V \int_{\text{inner cylinder}} \sigma' \, dA$$

Thus, we can get the amount of charge induced on the inner cylinder by the point charge between the plates:

$$\int_{\text{inner cylinder}} \sigma \, dA - \int_{\text{inner cylinder}} \sigma' \, dA = \text{charge induced on inner cylinder} = \frac{\Phi(c)}{V}$$

and calculating the right hand side is very easy. Thus as the charge moves between the plates, you can easily determine, as a function of time, the current that flows in an external circuit maintaining the potential.

5.Jackson, Ch. 2 problem 2.1, subsectons a, b, c, d, f.

A point charge θ

Let the infinite grounded conductor be in the $z = 0$ plane and let the charge q be on the $z -$ axis at $z = d > 0$. Then we can calculate the field in the $z \geq 0$ region as the sum of the field due to q itself and the field of a charge $-q$ located on the $z -$ axis at $z = -d$. It is very important to notice that the field due to the "image" charge is not due to an actual charge at the indicated image position, but is the cumulative effect of the distributed induced charge on the conducting plate. This charge has moved in from infinity (in our idealization) as the exciting charge q was moved into place.

The potential at the point $(z \geq 0,\ \rho, \varphi) = (x = \rho \cos \varphi.\ y = \rho \sin \varphi,\ z \geq 0)$ is

In[8]:= $\Phi[z\_, \rho\_] := \frac{1}{4 \pi \varepsilon_0} \left(\frac{q}{\sqrt{\rho^2 + (z - d)^2}} - \frac{q}{\sqrt{\rho^2 + (z + d)^2}} \right)$

It satisfies the boundary conditions $\Phi(z = 0, \rho, \varphi) = 0$, $\Phi = 0$ at ∞, and in the region $z \geq 0$ it satisfies Poisson's equation for a point charge at $(\rho = 0,\ z = d)$ and so by the uniqueness theorem, it is the potential in the region $z \geq 0$.

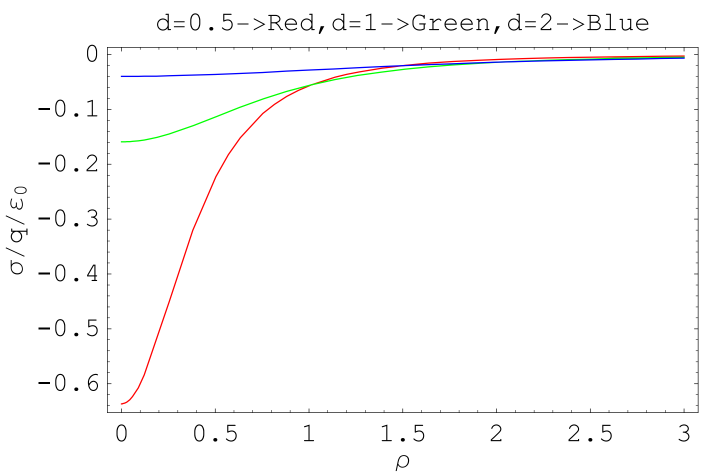
a) The surface charge density $\sigma(\rho)$ on the plate satisfies $\frac{\sigma(\rho)}{\varepsilon_0} = -\partial_z \Phi(z, \rho)|_{z=0}$.

In[9]:= $\sigma[\rho\_] := -\varepsilon_0 \, (\partial_z \Phi[z, \rho]) \, /. \, z \to 0$

In[10]:= $\sigma[\rho]$

Out[10]= $-\frac{d\, q}{2 \pi (d^2 + \rho^2)^{3/2}}$

In[11]:=
```
Plot[Evaluate[{σ[ρ] /. {d → .5},
    σ[ρ] /. {d → 1},
    σ[ρ] /. {d → 2}} /. {
    q → 1, ε0 → 1}], {ρ, 0, 3}, PlotRange → All,
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]},
 Frame → True, PlotLabel -> "d=0.5->Red,d=1->Green,d=2->Blue",
 FrameLabel → {"ρ", "σ/q/ε0"}]
```



Out[11]= - Graphics -

It is interesting to find out how much charge is induced on the plate (the integral is elementary).

In[12]:= $\text{Simplify}\left[2\pi\int_0^\infty \sigma[\rho]\,\rho\,d\rho,\ d>0\right]$

Out[12]= $-q$

Of course, it is just the value of the image charge.

b) The force on the charge q is in the $-z$ direction and its magnitude is

In[13]:= $\text{Fq} = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{(2\,d)^2}$

Out[13]= $\frac{q^2}{16\,d^2\,\pi\,\varepsilon_0}$

Alternatively, calculate the force (in the $+z$ direction) by integrating over the contributions of all of the induced charge on the plate

In[14]:= $\text{Fq2} = \frac{2\pi q}{4\pi\varepsilon_0}\int_0^\infty \sigma[\rho]\,\frac{d}{(d^2+\rho^2)^{3/2}}\,\rho\,d\rho$

Out[14]= $-\frac{q^2}{16\,d^2\,\pi\,\varepsilon_0}$

Of course, this yields the same result.

c) Alternatively, we can calculate the force acting on the plane using the fact that the force on an element of surface charge $\sigma\, dA$ is $\frac{\sigma^2}{2\,\varepsilon_0}\, dA$ acting normal to the surface and away from it. Thus the force on the plate is in the $+z$ direction and has magnitude

In[15]:= $$\text{Fplate} = \frac{2\,\pi}{2\,\varepsilon_0} \int_0^\infty \sigma[\rho]^2\, \rho\, d\rho$$

Out[15]= $$\frac{q^2}{16\, d^2\, \pi\, \varepsilon_0}$$

which is of course the expected value, being equal and opposite to the force on the charge.

The factor of 2 in the expression for the outward stress on a charged conductor seems a bit mysterious since one is tempted to write the force without it using Coulomb's law. However, the electric field here is due to the charge density itself, and a little more care is necessary. Here is how to get the result.

Think of the surface charge as being actually distributed slightly into the conductor and not a mathematical delta function on its surface. Then letting z be distance into the conductor, $z = 0$ at the surface, we can write from $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ that $\varepsilon_0\, \frac{dE_z}{dz} = \rho(z)$. The force on the conductor due to the charge in $dz\, dA$ is $dF_z = dA\, E_z(z)\, \rho(z)\, dz = dA\, \varepsilon_0\, E_z(z)\, \frac{dE_z}{dz}\, dz = dA\, \frac{1}{2}\, \varepsilon_0\, d(E_z(z)^2)$. Thus the total force per unit area is $\int_0^\infty \frac{1}{2}\, \varepsilon_0\, d(E_z(z)^2) = \frac{1}{2}\, \varepsilon_0\, E_{\text{surface}}^2 = \frac{1}{2\,\varepsilon_0}\, \sigma^2$.

d) The work necessary to move the charge from z to $z + dz$ is $dW = \frac{q^2\, dz}{16\, z^2\, \pi\, \varepsilon_0}$. The work to move it from $z = d$ to ∞ is

In[16]:= $$\int_d^\infty \frac{q^2}{16\, z^2\, \pi\, \varepsilon_0}\, dz$$

Out[16]= $$\frac{q^2}{16\, d\, \pi\, \varepsilon_0}$$

f) To get the work in electron volts, we set q to be the electron charge, e, then divide out one factor of e to become the "electron" in electron-Volt, and then evaluate in SI units producing Volts.

In[17]:= $$\frac{e}{16\, d\, \pi\, \varepsilon_0} \;/.\; \left\{e \to 1.6\; 10^{-19},\; d \to 10^{-10},\; \varepsilon_0 \to \frac{1}{9\; 10^{16}\; 4\, \pi\, 10^{-7}}\right\}$$

Out[17]= 3.6

In[18]:=

So the result is 3.6 eV.

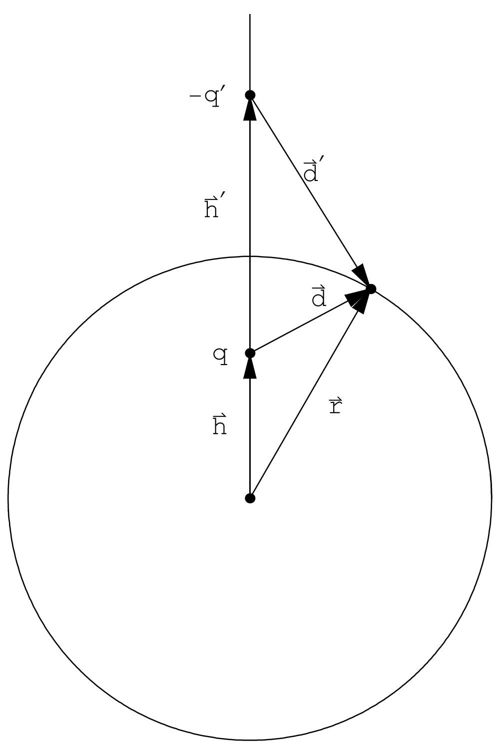
6. Jackson, Ch. 2 problem 2.2.

For a point charge at radius h on the $z-$axis inside a grounded sphere of radius a, $0 < h < a$, we consider an image charge $-q'$ at radius $h' > a$ on the $z-$axis. Then we adjust the two parameters q' and h' so that the sphere at radius a is at zero potential (if possible - it is not *a priori* obvious that this will work!)

Start graphics definitions

End graphics definitions

```
In[20]:= Show[sphericalimage, AspectRatio → 3 / 2]
```




```
Out[20]= - Graphics -
```

Notice that $|\vec{r}| = a$.

The potential at point $\vec{r}$ is

In[21]:= $\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d} - \frac{q'}{d'}\right)$

Out[21]= $\frac{\frac{q}{d} - \frac{q'}{d'}}{4\pi\varepsilon_0}$

But $d = \left|\vec{r} - \vec{h}\right| = |a\,\vec{e}_r - h\,\vec{e}_z|$ and $d' = \left|\vec{r} - \vec{d}'\right| = |a\,\vec{e}_r - h'\,\vec{e}_z|$. So we have

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\frac{a}{q}\vec{e}_r - \frac{h}{q}\vec{e}_z|} - \frac{1}{|\frac{a}{q'}\vec{e}_r - \frac{h'}{q'}\vec{e}_z|}\right)$$

Now use the fact that $|a\,\vec{u} + b\,\vec{v}| = |b\,\vec{u} + a\,\vec{v}| = \sqrt{a^2 + b^2 + 2\,a\,b\,\vec{u}\cdot\vec{v}}$ if $\vec{u}$ and $\vec{v}$ are unit vectors. So we can write

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\frac{a}{q}\vec{e}_r - \frac{h}{q}\vec{e}_z|} - \frac{1}{|\frac{h'}{q'}\vec{e}_r - \frac{a}{q'}\vec{e}_z|}\right)$$

and this will vanish if $a/q = h'/q'$ and $h/q = a/q'$ or $q' = \frac{a}{h}\,q$ and $h' = a\,\frac{q'}{q} = \frac{a^2}{h}$.

a) The potential at a point $\vec{r} =$ inside the sphere is

In[22]:= $\Phi[r\_, \theta\_] := \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + h^2 - 2\,r\,h\,\text{Cos}[\theta]}} - \frac{a}{h}\frac{1}{\sqrt{r^2 + a^4/h^2 - 2\,r\,\frac{a^2}{h}\,\text{Cos}[\theta]}}\right)$

Check for typing errors by checking the boundary condition that the sphere of radius a is at zero potential.

In[23]:= `Simplify[Φ[a, θ], {a > 0, h > 0}]`

Out[23]= 0

b) The induced surface charge on the inside surface of the grounded sphere is

In[24]:= $\sigma[\theta\_] := \varepsilon_0\,(\partial_r\,\Phi[r, \theta])\,/.\,r \to a$

In[25]:= `Simplify[σ[θ], {a > 0, h > 0}]`

Out[25]= $\frac{(-a^2 + h^2)\,q}{4\,a\,\pi\,(a^2 + h^2 - 2\,a\,h\,\text{Cos}[\theta])^{3/2}}$

Check the trivial case that $h \to 0$.

In[26]:= `Simplify[% /. h → 0, {a > 0, h > 0}]`

Out[26]= $-\frac{q}{4\,a^2\,\pi}$

More generally, check the integral of the surface charge density, which should be $-q$.

In[27]:= $\text{Simplify}\left[\int_0^\pi \sigma[\theta]\,2\,\pi\,a^2\,\text{Sin}[\theta]\,d\theta,\ \{a > h,\ h > 0\}\right]$

Out[27]= $-q$

c) Clearly we can get the force on q from the image charge (since the cumulative effect of the surface charge on the inside of the sphere is just the field on the inside of the sphere produced by the image), so it is along the $+z-$ axis with magnitude

$$F = \frac{q^2}{4\pi\varepsilon_0}\frac{a}{h}\frac{1}{(h'-h)^2} = \frac{q^2}{4\pi\varepsilon_0}\frac{a\,h}{(a^2-h^2)^2} = \frac{q^2}{4\pi\varepsilon_0}\frac{a\,h}{(a-h)^2\,(a+h)^2}$$

which goes to zero as $h\to 0$and to $\frac{q^2}{4\pi\varepsilon_0}\frac{1}{4\,(a-h)^2}$as $h\to a$, which is the plane sheet case. We can also get the result by integrating over the charge density to get the force in the $+z$ direction. To help *Mathematica* do the integral in reasonable time, make the substitution $\mathrm{Cos}[\theta]\to x$ and do a bit of pre-simplification on the integrand.

In[28]:=
```
Simplify[
 -q/(4 π ε0) 2 π ∫_{-1}^{1} Evaluate[Simplify[σ[θ] /. Cos[θ] → x, {a > h, h > 0}]/d^2 (a x - h)/d /.
     d -> Sqrt[a^2 + h^2 - 2 a h x]] a^2 dx, {a > h, h > 0}]
```

Out[28]=
$$\frac{a\,h\,q^2}{4\,(a-h)^2\,(a+h)^2\,\pi\,\varepsilon_0}$$

As above.

d) If the sphere is set to potential V then, inside it, the potential is what was calculated in a) above plus V. Notice that this satisfies the boundary condition that the inside surface is an equipotential V and the Poisson equation is satisfied inside, and, being A solution it is THE solution by the uniqueness theorem.

Similarly, if a charge Q is added to the conductor, it becomes an equipotential V' and so inside the potential is a) above plus V'. The value of $V'$is determined by the value of Q (all of which is on the outside surface) and the shape of the *OUTSIDE* surface of the conductor. In particular, if the outside conducting surface has a capacitance C with respect to infinity, then we have

$V' = \frac{Q}{C}$.

For example, if the outside is a spherical surface of radius a' then $V' = \frac{Q}{4\pi\varepsilon_0\,a'}$.