Ph196bEM Homework2 Due: Friday, Feb. 27, 5pm, 337 Lauritsen

7. Jackson, problem 2.1, section e.

Be sure to carefully handle the singularity due to the point charge in this problem. In particular, even when the point charge has been removed to infinity (and there is no induced charge on the grounded plate), there is still an energy in the field around the point charge, exactly equal to the corresponding infinite term in the energy when the point charge is near the plate. This energy is formally infinite, but not physically so since other interactions than E&M must be taken into account. So to relate stored energy to work done in making a change, you must subtract the initial and final electrostatically stored energies.

8. Jackson problem 1.15 (Thomson's Theorem).

Note that what is given is the charge Q_i to be placed on conductor i, not just the total charge $\sum_i Q_i$ to be placed on all of the conductors.

Hint: Consider the integral

$$\int_V \left(\vec{E}^2 - \vec{E}'^2\right) d^3 r = \int_V \left(\left(\vec{E} - \vec{E}'\right)^2 + 2\,\vec{E}\cdot\vec{E}' - 2\,\vec{E}'^2\right) d^3 r$$

where $\vec{E}$ is the field produced by the arbitrarily given distribution of potentials on the surfaces and $\vec{E}'$, that produced when each surface is an equipotential, i.e., the surfaces are conductors. Of course, use the relation

$$\vec{\nabla}\cdot\left(\psi\,\vec{\nabla}\phi\right) = \vec{\nabla}\psi\cdot\vec{\nabla}\phi + \psi\,\vec{\nabla}^2\phi$$

as appropriate.

9. Cylindrical coordinates are defined by

$x = \rho\cos\varphi,\ \ y = \rho\sin\varphi,\ \ z = z,$ where $\rho \in [0, \infty),\ \ \varphi \in [0, 2\pi),\ \ z \in (-\infty, \infty).$

Find the divergence and Laplacian in this coordinate system (answer on back cover of Jackson).

10. Define oblate ellipsoidal coordinates (μ, ν, φ), with parameter $R > 0$, by

$x = R\cosh\mu\,\sin\nu\,\cos\varphi$

$y = R\cosh\mu\,\sin\nu\,\sin\varphi$

$z = R\sinh\mu\,\cos\nu$

with $\mu \in \{0, \infty),\ \ \nu \in [0, \pi],\ \ \varphi \in [0, 2\pi).$

a) Make a sketch of the lines of constant μ and ν in a fixed φ plane.

b) Show that these coordinates are orthogonal and right handed in the order

(μ, ν, φ), and find the scale factors $(h_\mu, h_\nu, h_\varphi)$. Identify the singular points of the transformation.

c) Finally, write out the gradient and use the electrostatics variational principle

to calculate the Laplacian and, from it and the gradient, get the divergence

operator in this coordinate system.

11. The principle of minimizing (or, more generally, finding a stationary point for) the action to find the electrostatic potential in a specified 3D volume V with specified boundary potentials on ∂V can be used in practice to find an approximate solution to the problem. The idea is to represent the potential in some way with a function that contains adjustable parameters, for any values of which the boundary conditions are satisfied. Then calculate the action as a function of the parameters, and finally choose that set of parameters that makes the action stationary; this will be the best possible approximation of the given form to the actual potential.

The extreme case of this strategy is to parametrize the potential with itself, i.e., with its values on the vertices of a rectangular grid, defined by a (small) regular spacing Δ in x, y, and z. Deform the boundary slightly, as necessary, to follow the 3D grid. The vertices can be parametrized by integers (i, j, k) so that the parameters are just $\Phi_{i,j,k}$ when (i, j, k) does **not** identify a point on the boundary. Then approximate $\left(\frac{\partial \Phi}{\partial x}\right)_{i.j.k} \simeq \frac{\Phi_{i+1,j,k} - \Phi_{i,j,k}}{\Delta}$ and similarly for the other two dimensions. Also take $\rho_{i,j,k}$ as the given charge density in V at the vertex (i, j, k). Finally, approximate the action integral by a sum and find the equations in the values of the potential that make it stationary. These equations can be solved iteratively by starting with, for example, zero potential on all the grid points inside V and the appropriate specified value on grid points on ∂V. This allows the calculation of a next approximation to the field. Then use it in the same equations, etc., and continue iterating until exhaustion calls a halt to the process (or more formally, the potential values become stable in value. In fact this procedure leads to a matrix equation and the problem becomes that of inverting a large sparse matrix. The solution by iteration can be shown to converge to the solution of the linear equations. This may seem pretty crude, but in fact it is quite practical and, with some embellishments, yields quite accurate solutions.

Find the approximate relaxation equations by minimizing the approximate action as indicated.