Ph196bEM Homework 2 Solutions - 2004

7. Jackson3ed, problem 2.1e

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

e) the potential energy between the charge q and its image [compare the answer to the work necessary to remove the charge from its position to infinity, and discuss].

Be sure to carefully handle the singularity due to the point charge in this problem. In particular, even when the point charge has been removed to infinity (and there is no induced charge on the grounded plate), there is still an energy in the field around the point charge, exactly equal to the corresponding infinite term in the energy when the point charge is near the plate. This energy is formally infinite, but not physically so since other interactions than E&M must be taken into account. So to relate stored energy to work done in making a change, you must subtract the initial and final electrostatically stored energies.

■ Solution

The electric field at any point in space is the sum of two parts, that due to the given point charge, $\vec{E}_q$, and that due to the charge distribution on the infinite grounded plate, $\vec{E}_{\text{plate}}$, in the $z = 0$ plane. Clearly, by symmetry, $\vec{E}_{\text{plate}}$ at points (x, y, z) and $(x, y, -z)$ differ only in that the component of the electric field normal to the plate has opposite signs. In particular, in $z > 0$, $\vec{E}_{\text{plate}}$ is the same as the field due to a point charge $-q$ located at $(0, 0, -d)$ whereas in $z < 0$, $\vec{E}_{\text{plate}}$ is the same as the field due to a point charge $-q$ located ar $(0, 0, +d)$ The electric field situation looks like the following.

Begin graphics

End graphics

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Show[fields,
 AspectRatio → 1, PlotRange → All,
 PlotLabel -> "Black: E due to q; Red: E due to plate"]
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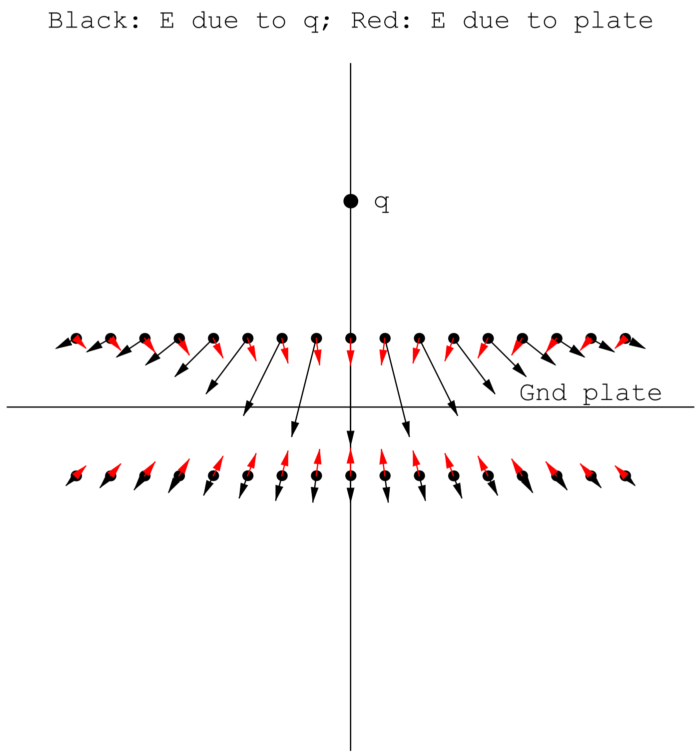

- Graphics -

The field due to the positive point charge q points away from q everywhere; the field due to the negative induced charge on the plate points generally toward the plate. Thus in the region $z \geq 0$ there is a net field whereas in $z \leq 0$ the two contributions are equal but oppositely directed so the total field is zero.

The energy stored in the electrostatic system when the point charge is near the plate is

$$\frac{\varepsilon_0}{2} \int_{\text{all space}} \left(\vec{E}_q + \vec{E}_{\text{plate}}\right)^2 d^3 r$$

and when the charge is moved a great distance from the plate,

$\frac{\varepsilon_0}{2} \int_{\text{all space}} (\vec{E}_q)^2 \, d^3 r.$

This last integral represents the energy necessary to build the point charge and is formally divergent in our theory (in the real world, other interactions in addition to E&M must be invoked to properly express the energy tied up in an electron, for example). To find out how much energy is required to bring the pre-existing point charge to the vicinity of the plate (or to remove it from the vicinity of the plate) we want the difference of the above two terms. Thus

$W =$ Energy to remove point charge from vicinity of plate =

$$\frac{\varepsilon_0}{2} \int_{\text{all space}} (\vec{E}_q)^2 \, d^3 r - \frac{\varepsilon_0}{2} \int_{\text{all space}} \left(\vec{E}_q + \vec{E}_{\text{plate}}\right)^2 d^3 r$$

or

$$W = \frac{-\varepsilon_0}{2} \int_{\text{all space}} \left(2 \, \vec{E}_q \cdot \vec{E}_{\text{plate}} + \vec{E}^2_{\text{plate}}\right) d^3 r.$$

This integral has a singularity at the position of the point charge in the integrand, but it's integral is well behaved as we will see shortly.

Let the charge be at $z = d > 0$ on the $z-$axis. Then

$$W = -\frac{\varepsilon_0}{2} \left(\frac{q}{4 \pi \varepsilon_0}\right)^2$$
$$\left(\int_{z>0} \left(-2 \, (\vec{r} - d \, \vec{e}_z) \cdot \frac{(\vec{r} + d \, \vec{e}_z)}{|\vec{r} - d \, \vec{e}_z|^3 \, |\vec{r} + d \, \vec{e}_z|^3} + \frac{(\vec{r} + d \, \vec{e}_z)^2}{|\vec{r} + d \, \vec{e}_z|^6}\right) d^3 r + \int_{z<0} \left(-2 \, \frac{(\vec{r} - d \, \vec{e}_z)^2}{|\vec{r} - d \, \vec{e}_z|^6} + \frac{(\vec{r} - d \, \vec{e}_z)^2}{|\vec{r} - d \, \vec{e}_z|^6}\right) d^3 r\right)$$

Now it is obvious that $\int_{z>0} \frac{(\vec{r} + d \, \vec{e}_z)^2}{|\vec{r} + d \, \vec{e}_z|^6} \, d^3 r = \int_{z<0} \frac{(\vec{r} - d \, \vec{e}_z)^2}{|\vec{r} - d \, \vec{e}_z|^6} \, d^3 r$. Therefore three of the terms in W cancel against one another leaving

$$W = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{z>0} 2\,\frac{(\vec{r}-d\,\vec{e}_z)\cdot(\vec{r}+d\,\vec{e}_z)}{|\vec{r}-d\,\vec{e}_z|^3\,|\vec{r}+d\,\vec{e}_z|^3}\,d^3 r$$
$$= \varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{z>0} \frac{r^2-d^2}{(x^2+y^2+(z-d)^2)^{3/2}\,(x^2+y^2+(z+d)^2)^{3/2}}\,d^3 r$$
$$= 2\pi\,\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{\rho=0}^{\infty}\int_{z=0}^{\infty} \frac{\rho^2+z^2-d^2}{(\rho^2+(z-d)^2)^{3/2}\,(\rho^2+(z+d)^2)^{3/2}}\,\rho\,d\rho\,dz$$
$$= \pi\,\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{x=0}^{\infty}\int_{z=0}^{\infty} \frac{x+z^2-d^2}{(x+(z-d)^2)^{3/2}\,(x+(z+d)^2)^{3/2}}\,dx\,dz \qquad \text{eq (1)}$$

For future reference, notice that

$$W = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{z>0} 2\,\frac{(\vec{r}-d\,\vec{e}_z)\cdot(\vec{r}+d\,\vec{e}_z)}{|\vec{r}-d\,\vec{e}_z|^3\,|\vec{r}+d\,\vec{e}_z|^3}\,d^3 r$$
$$= -\varepsilon_0 \int_{z>0} \vec{E}_{\text{due to } q}\cdot\vec{E}_{\text{due to plate}}\,d^3 r \qquad \text{eq (2).}$$

Now do the integral in eq (1).

```
Simplify[π ε0 (q/(4 π ε0))^2 ∫_0^∞ (x + z^2 - d^2) / ((x + (z - d)^2)^(3/2) (x + (z + d)^2)^(3/2)) dx,
  {d > 0, z > 0, d ≠ z}]
```

$$\frac{q^2\,(-d+z+\text{Abs}[d-z])}{32\,\pi\,z^2\,\text{Abs}[d-z]\,\varepsilon_0}$$

So being careful about the absolute values (which arose from $\sqrt{x^2} = |x|$ and which makes this term vanish for $z < d$) we get

```
W = ∫_d^∞ q^2 (-d + z - d + z) / (32 π z^2 (z - d) ε0) dz
```

$$\frac{q^2}{16\,d\,\pi\,\varepsilon_0}$$

This, as it must be, is the same result obtained by directly calculating the work by using the forces acting on the charge being moved.

This frontal attack on the problem yields the correct answer and *Mathematica* is smart enough to know that $\sqrt{x^2} = |x|$ but it is very tempting when doing the integrations by hand to make the error $\sqrt{x^2} = x$, which gives the wrong answer. An alternative approach which does not tempt you to fall in this trap for the unwary is to use Gauss's law to undo part of the machinery leading to the result that energy is the integral of E^2. We had obtained above

$$W = -\varepsilon_0 \int_{z>0} \vec{E}_{\text{due to q}} \cdot \vec{E}_{\text{due to plate}} \, d^3 r$$
$$= -\varepsilon_0 \int_{z>0} \vec{\nabla} \Phi_{\text{due to } q} \cdot \vec{\nabla} \Phi_{\text{due to plate}} \, d^3 r$$

By Green's relation we can write

$$\vec{\nabla} \cdot \left(\Phi_{\text{due to plate}} \vec{\nabla} \Phi_{\text{due to } q}\right)$$
$$= \vec{\nabla} \Phi_{\text{due to plate}} \cdot \vec{\nabla} \Phi_{\text{due to } q} + \Phi_{\text{due to plate}} \nabla^2 \Phi_{\text{due to } q}$$
$$= \vec{\nabla} \Phi_{\text{due to } q} \cdot \vec{\nabla} \Phi_{\text{due to plate}} - \frac{q}{\varepsilon_0} \Phi_{\text{due to plate}} \delta^3(\vec{r} - d\, \vec{e}_z)$$

So

$$W = -\varepsilon_0 \int_{z>0} \left(\vec{\nabla} \cdot \left(\Phi_{\text{due to plate}} \vec{\nabla} \Phi_{\text{due to } q}\right) + \frac{q}{\varepsilon_0} \Phi_{\text{due to plate}} \delta^3(\vec{r} - d\, \vec{e}_z)\right) d^3 r$$
$$= \varepsilon_0 \int_{z=0} \Phi_{\text{due to plate}} \vec{E}_{\text{due to } q} \cdot d\vec{A} - q\, \Phi_{\text{due to plate}}(\vec{r} = d\, \vec{e}_z).$$

Note that the singularity on the point charge has been defanged.
So

$$W = \varepsilon_0 \int_{\rho=0}^{\infty} \left(\frac{-q}{4\pi\varepsilon_0} \frac{1}{\sqrt{d^2+\rho^2}}\right)\left(\frac{q}{4\pi\varepsilon_0} \frac{d}{(d^2+\rho^2)^{3/2}}\right) 2\pi\rho\, d\rho + q \frac{q}{4\pi\varepsilon_0} \frac{1}{2d}.$$

Let $x = \rho^2$. Then

$$-\pi\, \varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_0^{\infty} \frac{d}{(x + d^2)^2} \, dx + \frac{q^2}{4\pi\varepsilon_0} \frac{1}{2\,d}$$

$$\frac{q^2}{16\, d\, \pi\, \varepsilon_0}$$

As we have above, but the integral is now trivial.

8. Jackson3ed, problem 1.15.

Prove *Thompson's Theorem:* If a number of conducting surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are so placed that every surface is an equipotential, as happens when they are conductors.

Note that what is given is the charge Q_i to be placed on conductor i, not just the total charge $\sum_i Q_i$ to be placed on all of the conductors.

Hint: Consider the integral

$$\int_V \left(\vec{E}^2 - \vec{E}'^2\right) d^3 r = \int_V \left(\left(\vec{E} - \vec{E}'\right)^2 + 2\,\vec{E}\cdot\vec{E}' - 2\,\vec{E}'^2\right) d^3 r$$

where $\vec{E}$ is the field produced by the arbitrarily given distribution of potentials on the surfaces and $\vec{E}'$, that produced when each surface is an equipotential, i.e., the surfaces are conductors. Of course, use the relation

$$\vec{\nabla}\cdot\left(\psi\,\vec{\nabla}\phi\right) = \vec{\nabla}\psi\cdot\vec{\nabla}\phi + \psi\,\vec{\nabla}^2\phi$$

as appropriate.

■ Solution

Let $\vec{E}$ be the electric field in V when the charges Q_i are placed on the boundary surfaces S_i in an arbitrary way and $\vec{E}'$, that when they are distributed in such a way that each surface is an equipotential, i.e., the surfaces are conductors. The geometry could look like this.

Begin graphics

End graphics

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In[2]:= Show[boundary, AspectRatio → 1, ImageSize → 72 3]
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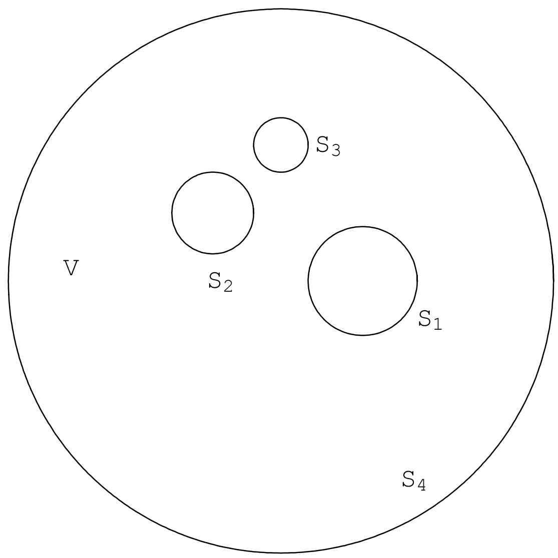


Out[2]= - Graphics -

Then what we want to show is that

$$\int_V E^2\, d^3 r - \int_V E'^2\, d^3 r \geq 0$$

when the charges on the surfaces are constrained to be as specified.

Use the identity

$$\int_V (E^2 - E'^2)\, d^3 r = \int_V \left(\left(\vec{E} - \vec{E}'\right)^2 + 2\,\vec{E}\cdot\vec{E}' - 2\,E'^2 \right) d^3 r.$$

With the corresponding potentials being Φ and Φ', we have

$$\int_V \left(\vec{E}\cdot\vec{E}' - E'^2 \right) d^3 r$$
$$= \int_V \left(\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi' - \vec{\nabla}\Phi'\cdot\vec{\nabla}\Phi' \right) d^3 r$$
$$= \int_V \vec{\nabla}\Phi'\cdot\vec{\nabla}(\Phi - \Phi')\, d^3 r$$
$$= \int_V \left(\vec{\nabla}\cdot\left(\Phi'\,\vec{\nabla}(\Phi - \Phi')\right) - \Phi'\,\nabla^2(\Phi - \Phi') \right) d^3 r$$

But in V, there are no free charges (except on the boundary and we can exclude the boundary from V). So we have $\nabla^2\,\Phi = \nabla^2\Phi' = 0$ and

$\int_V \left(\vec{E} \cdot \vec{E}' - E'^2 \right) d^3 r = \int_V \vec{\nabla} \cdot \left(\Phi' \vec{\nabla} (\Phi - \Phi') \right) d^3 r = \int_{\partial V} \Phi' \vec{\nabla} (\Phi - \Phi') \cdot d\vec{A}.$

But the charges were assumed to be arranged in just such a way that Φ' is constant on each part of the bounding surface and furthermore the values of the charges to be distributed on each surface are specified so that $\int_{S_i} \vec{\nabla} (\Phi - \Phi') \cdot d\vec{A} = 0$ since the charge density on the surface is just proportional to $E_\perp$. Thus we get

$\int_V \left(\vec{E} \cdot \vec{E}' - E'^2 \right) d^3 r = \sum_i \Phi'_i \int_{S_i} \vec{\nabla} (\Phi - \Phi') \cdot d\vec{A} = 0.$

So finally

$$\int_V (E^2 - E'^2)\, d^3 r = \int_V \left(\vec{E} - \vec{E}' \right)^2 d^3 r \geq 0$$

with equality if and only if $\vec{E} = \vec{E}'$. So we have Thomson's remarkable result that conductors arrange the charges on their surfaces to both make all the surfaces equipotentials, but also to absolutely minimize the electrostatically stored energy, for specified charges on the conductors.

9. Field Equation in Cylindrical Coordinates

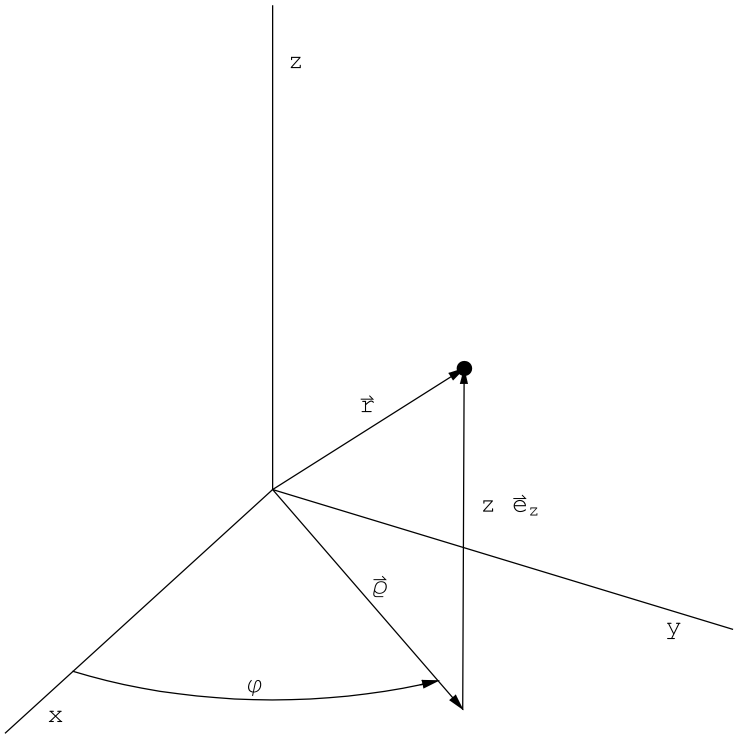
Cylindrical coordinates are defined by

$x = \rho \cos \varphi, \ y = \rho \sin \varphi, \ z = z,$ where $\rho \in [0, \infty), \ \varphi \in [0, 2\pi), \ z \in (-\infty, \infty).$

Find the divergence and Laplacian in this coordinate system (answer on back cover of Jackson).

■ Solution

a) Cylindrical coordinates are defined pictorially like this:



You can derive the field equation from the action principle, expressing the gradient in cylindrical coordinates. From $x = \rho\cos\varphi,\ \ y = \rho\sin\varphi,\ \ z = z$ we easily get the Jacobian matrix:

$$\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The scale factors are just the lengths of the vectors represented by each of the three columns of this matrix. So,

$h_\rho = 1,\ \ h_\varphi = \rho,\ \ h_z = 1,$

and the gradient of a scalar Φ is

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial\rho}\,\vec{u}_\rho + \frac{1}{\rho}\,\frac{\partial\Phi}{\partial\varphi}\,\vec{u}_\varphi + \frac{\partial\Phi}{\partial z}\,\vec{u}_z.$$

From the Jacobian matrix, the unit vectors can be expressed in cartesian coordinates as

$\vec{u}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y,$

$\vec{u}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y,$
$\vec{u}_z = \vec{e}_z.$

and so we can formally check right handedness:

$\vec{u}_\rho \times \vec{u}_\varphi = (\cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y) \times (-\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y)$

$= (\cos^2\varphi + \sin^2\varphi)\, \vec{e}_x \times \vec{e}_y$

$= \vec{e}_z = \vec{u}_z.$

The spatial volume element is

$d^3 r = dx\, dy\, dz = d\rho\ \rho\, d\varphi\ dz = \rho\, d\rho\, d\varphi\ dz = \frac{1}{2}\, d\rho^2\, d\varphi\ dz$

and the action is

$\mathcal{A} = \int \left(\frac{1}{2}\, \varepsilon_0 \left(\left(\frac{\partial \Phi}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\ \frac{\partial \Phi}{\partial \varphi}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) - \rho_{\text{charge}}\, \Phi\right) \rho\ \ d\rho\, d\varphi\, dz$

Thus since in the derivation of the field equations from the action principle we used

$\mathcal{A} = \int \mathcal{L}\left(\frac{\partial \Phi}{\partial q_i}, \Phi, q_i\right)\ dq_1\, dq_2\, dq_3$

we get in the case in hand

$\mathcal{L} = \left(\frac{1}{2}\, \varepsilon_0 \left(\left(\frac{\partial \Phi}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\ \frac{\partial \Phi}{\partial \varphi}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) - \rho_{\text{charge}}\, \Phi\right) \rho\ \ .$

The subtle point is that the action principle is formulated in an integral over the parameter space, not geometric space so the Jacobian associated with the transformation from cartesian coordinates to new parameters gets incorporated into the Lagrangian density.

Finally, the field equation is

$\frac{\partial}{\partial \rho}\left(\varepsilon_0\, \rho\, \frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \varphi}\left(\varepsilon_0\, \rho\, \frac{1}{\rho^2}\ \frac{\partial \Phi}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(\varepsilon_0\, \rho\, \frac{\partial \Phi}{\partial z}\right) - (-\rho\, \rho_{\text{charge}}) = 0,$

or

$\frac{1}{\rho}\ \frac{\partial}{\partial \rho}\left(\rho\, \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\ \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = -\frac{\rho_{\text{charge}}}{\varepsilon_0}\,.$

This is of course the expression for $\nabla^2 \Phi = -\frac{\rho_{\text{charge}}}{\varepsilon_0}$ in cylindrical coordinates. You can use $\nabla^2 \Phi = \vec{\nabla} \cdot \vec{\nabla} \Phi$ to get the expression for $\vec{\nabla} \cdot \vec{A}$ in cylindrical coordinates by just replacing the components of $\vec{\nabla} \Phi$, expressed in the curvilinear coordinate basis at a point (the components along $\vec{u}_{r_i}$) with the components of $\vec{A}$ resolved in this basis. The result is easy to see to be

$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\ \frac{\partial}{\partial \rho}(\rho\, A_\rho) + \frac{1}{\rho}\ \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z},$

as agrees with the back of Jackson. You should notice the dimensions in each of the terms in these expressions: each term in $\nabla^2 \Phi$ has dimension of Φ/length^2 and each component of $\vec{\nabla} \cdot \vec{A}$ has units of A/length. You can use this little check to notice accidental errors in using these forms.

10. Oblate Ellipsoidal Coordinates

Define oblate ellipsoidal coordinates (μ, ν, φ), with parameter R > 0, by

$x = R \cosh \mu \sin \nu \cos \varphi$

$y = R \cosh \mu \sin \nu \sin \varphi$

$z = R \sinh \mu \cos \nu$

with $\mu \in \{0, \infty)$, $\nu \in [0, \pi]$, $\varphi \in [0, 2\pi)$.

a) Make a sketch of the lines of constant μ and ν in a fixed φ plane.

b) Show that these coordinates are orthogonal and right handed in the order

(μ, ν, φ), and find the scale factors $(h_\mu, h_\nu, h_\varphi)$. Identify the singular points of the transformation.

c) Finally, write out the gradient and use the electrostatics variational principle

to calculate the Laplacian and, from it and the gradient, get the divergence

operator in this coordinate system.

■ Solution

b) Oblate ellipsoidal coordinates are defined by

$x = R \cosh \mu \;\; \sin \nu \;\cos \varphi$
$y = R \cosh \mu \;\; \sin \nu \;\sin \varphi$
$z = R \sinh \mu \;\cos \nu$

with $R > 0$ a given parameter, $\mu \geq 0$, $0 \leq \nu \leq \pi$, $0 \leq \varphi < 2\pi$. Note that as in spherical coordinates where $\sin \theta \geq 0$, here we have $\sin \nu \geq 0$.

First, the name of these coordinates is oblate spheroidal (or ellipsoidal); an oblate spheroid is sort of the shape of a basketball when you sit on it. A prolate spheroid is like a football. The ellipsoids in the coordinates given here are like a flattened sphere; interchange the "cosh" and "sinh" to get the prolate case. First the range of μ is 0 to ∞, that of ν is 0 to π, and that of φ is 0 to 2π. Draw the coordinates for the plane $\varphi = 0$, so $x = R \cosh \mu \;\sin \nu$, $z = R \sinh \mu \;\cos \nu$. Set $R = 1$. This is easy in *Mathematica* (I have hidden the details of the definition of the graphics).

Begin graphics

End graphics

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In[28]:= Show[pltfixedu, Graphics[Dashing[{.02, .02}]], pltfixedv,
           AspectRatio -> 2, ImageSize → 3 72, DisplayFunction → $DisplayFunction]
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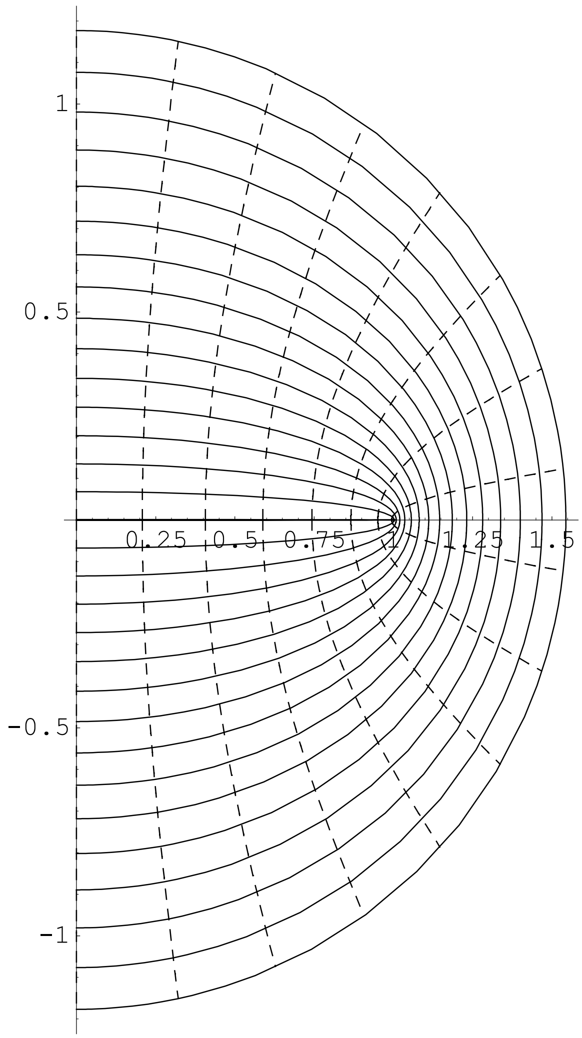


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Out[28]=  - Graphics -
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The solid lines have fixed values of μ and the dashed ones, fixed ν. Note that you can write

$\left(\frac{x}{R\cosh\mu}\right)^2 + \left(\frac{z}{R\sinh\mu}\right)^2 = \sin^2\nu + \cos^2\nu = 1,$

which defines ellipses as the solid lines are. Similarly

$\left(\frac{x}{R\sin\nu}\right)^2 - \left(\frac{z}{R\cos\nu}\right)^2 = \cosh^2\mu - \sinh^2\mu = 1,$

which defines hyperbolas, as the dotted lines are.

The plot is only in the $x \geq 0$ region since $\varphi = 0$, $\cosh \mu > 0$, and $\sin \nu \geq 0$; the $x \leq 0$ region is covered by $\varphi = \pi$. The disk in the $x - z$ plane of radius R is special, since it is covered by a constant value 0 of the μ coordinate. Note that $(\mu, \nu) = (0, 0)$ covers the top of this disk while $(\mu, \nu) = (0, \pi)$ covers its bottom. Further, this coordinate system is like ordinary spherical polar coordinates in which the origin point has been smashed into a disk and we can distinguish one side of the disk from the other.

The vectors $\left(\frac{d\vec{r}}{d\mu}, \frac{d\vec{r}}{d\nu}, \frac{d\vec{r}}{d\varphi}\right)$ can be written as $(h_\mu \vec{e}_\mu,\ h_\nu \vec{e}_\nu,\ h_\varphi \vec{e}_\varphi)$, where the three vectors are unit length. Thus $h_\mu = \left| \frac{d\vec{r}}{d\mu} \right| = \sqrt{\left(\frac{d\vec{r}}{d\mu}\right)^2}$, $h_\nu = \left| \frac{d\vec{r}}{d\nu} \right| = \sqrt{\left(\frac{d\vec{r}}{d\nu}\right)^2}$, $h_\varphi = \left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\left(\frac{d\vec{r}}{d\varphi}\right)^2}$. Writing the three vectors as columns, the form $\left(\frac{d\vec{r}}{d\mu}, \frac{d\vec{r}}{d\nu}, \frac{d\vec{r}}{d\varphi}\right)$ is just the Jacobian matrix,

$$\underset{\approx}{J} = \begin{pmatrix} \frac{dx}{d\mu} & \frac{dx}{d\nu} & \frac{dx}{d\varphi} \\ \frac{dy}{d\mu} & \frac{dy}{d\nu} & \frac{dy}{d\varphi} \\ \frac{dz}{d\mu} & \frac{dz}{d\nu} & \frac{dz}{d\varphi} \end{pmatrix} = \underset{\approx}{Q}\,\underset{\approx}{D} = \begin{pmatrix} \frac{1}{h_\mu}\frac{dx}{d\mu} & \frac{1}{h_\nu}\frac{dx}{d\nu} & \frac{1}{h_\varphi}\frac{dx}{d\varphi} \\ \frac{1}{h_\mu}\frac{dy}{d\mu} & \frac{1}{h_\nu}\frac{dy}{d\nu} & \frac{1}{h_\varphi}\frac{dy}{d\varphi} \\ \frac{1}{h_\mu}\frac{dz}{d\mu} & \frac{1}{h_\nu}\frac{dz}{d\nu} & \frac{1}{h_\varphi}\frac{dz}{d\varphi} \end{pmatrix} \begin{pmatrix} h_\mu & 0 & 0 \\ 0 & h_\nu & 0 \\ 0 & 0 & h_\varphi \end{pmatrix}.$$

For the coordinate system in hand,

$$\frac{d\vec{r}}{d\mu} = \begin{pmatrix} R \sinh\mu \ \sin\nu \ \cos\varphi \\ R \sinh\mu \ \sin\nu \sin\varphi \\ R \cosh\mu \ \cos\nu \end{pmatrix} \Rightarrow h_\mu = R \sqrt{\sinh^2\mu \ \sin^2\nu + \cosh^2\mu \ \cos^2\nu}$$

$$= R \sqrt{\sinh^2\mu \ \sin^2\nu + \cosh^2\mu - \cosh^2\mu \ \sin^2\nu}$$

$$= R \sqrt{\cosh^2\mu - \sin^2\nu}.$$

Notice that $\cosh^2\mu \geq 1$ and $\sin^2\nu \leq 1$. Thus h_μ is real and positive except at the singular point $\mu = \nu = 0$.

$$\frac{d\vec{r}}{d\nu} = \begin{pmatrix} R \cosh\mu \ \cos\nu \ \cos\varphi \\ R \cosh\mu \ \cos\nu \sin\varphi \\ -R \sinh\mu \ \sin\nu \end{pmatrix} \Rightarrow h_\nu = R \sqrt{\cosh^2\mu \ \cos^2\nu + \sinh^2\mu \ \sin^2\nu}$$

$$= R \sqrt{\cosh^2\mu - \sin^2\nu} = h_\mu.$$

Finally,

$$\frac{d\vec{r}}{d\varphi} = \begin{pmatrix} -R \cosh\mu \sin\nu \ \sin\varphi \\ R \cosh\mu \ \sin\nu \cos\varphi \\ 0 \end{pmatrix} \Rightarrow h_\varphi = R \cosh\mu \sin\nu \ \text{(note that I have used } \sin\nu \geq 0 \text{ and } \cosh\mu > 0\text{)}.$$

Now write the matrix $\underset{\approx}{Q}$ and show that it is orthogonal with a unit determinate which is just $\vec{e}_\mu \cdot (\vec{e}_\nu \times \vec{e}_\varphi)$. This will show that the system is a right-handed orthogonal curvilinear coordinate system (with the order of the unit vectors as specified).

In[29]:=
```
Om = {{Sinh[μ] Sin[v] Cos[φ]/Sqrt[Cosh[μ]^2-Sin[v]^2], Cosh[μ] Cos[v] Cos[φ]/Sqrt[Cosh[μ]^2-Sin[v]^2], -Cosh[μ] Sin[v] Sin[φ]/(Cosh[μ] Sin[v])},
      {Sinh[μ] Sin[v] Sin[φ]/Sqrt[Cosh[μ]^2-Sin[v]^2], Cosh[μ] Cos[v] Sin[φ]/Sqrt[Cosh[μ]^2-Sin[v]^2], Cosh[μ] Sin[v] Cos[φ]/(Cosh[μ] Sin[v])},
      {Cosh[μ] Cos[v]/Sqrt[Cosh[μ]^2-Sin[v]^2], -Sinh[μ] Sin[v]/Sqrt[Cosh[μ]^2-Sin[v]^2], 0}};
```

Check the orthogonality and the determinate of this matrix

In[30]:=
```
Simplify[Transpose[Om].Om]
```

Out[30]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

In[31]:=
```
Simplify[Det[Om]]
```

Out[31]= 1

So we have a nice coordinate system, i.e., the unit vectors are mutually orthogonal. This formally shows that the lines of fixed μ and v intersect orthogonally in the diagram above. Further from the equation for the Jacobian of the transformation from cartesian to oblate spheroidal coordinates we have

$$J = \left|\det \underset{\approx}{J}\right| = \left|\det\left(\underset{\approx}{Q}\,\underset{\approx}{D}\right)\right| = \left|\det \underset{\approx}{Q}\right| \left|\det \underset{\approx}{D}\right| = h_\mu\, h_v\, h_\varphi = R^3\,(\cosh^2\mu - \sin^2 v)\cosh\mu\sin v.$$

The gradient is

$$\vec{\nabla}\Phi = \frac{1}{R\sqrt{\cosh^2\mu - \sin^2 v}}\,\frac{\partial\Phi}{\partial\mu}\,\vec{e}_\mu + \frac{1}{R\sqrt{\cosh^2\mu - \sin^2 v}}\,\frac{\partial\Phi}{\partial v}\,\vec{e}_v + \frac{1}{R\cosh\mu\sin v}\,\frac{\partial\Phi}{\partial\varphi}\,\vec{e}_\varphi.$$

The Lagrangian density is the coefficient of $d\mu\, dv\, d\varphi$ in the action integral and so is

$$\mathcal{L} = \left(\frac{\varepsilon_0}{2}\left(\left(\frac{1}{R\sqrt{\cosh^2\mu - \sin^2 v}}\,\frac{\partial\Phi}{\partial\mu}\right)^2 + \left(\frac{1}{R\sqrt{\cosh^2\mu - \sin^2 v}}\,\frac{\partial\Phi}{\partial v}\right)^2 + \left(\frac{1}{R\cosh\mu\sin v}\,\frac{\partial\Phi}{\partial\varphi}\right)^2\right) - \rho\,\Phi\right) R^3\,(\cosh^2\mu - \sin^2 v)\cosh\mu\sin v$$

$$= \left(\frac{\varepsilon_0}{2}\,R\left(\frac{1}{\cosh^2\mu - \sin^2 v}\left(\left(\frac{\partial\Phi}{\partial\mu}\right)^2 + \left(\frac{\partial\Phi}{\partial v}\right)^2\right) + \frac{1}{\cosh^2\mu\sin^2 v}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\right) - R^3\,\rho\,\Phi\right)(\cosh^2\mu - \sin^2 v)\cosh\mu\sin v$$

$$= R\,\frac{\varepsilon_0}{2}\left(\cosh\mu\sin v\left(\left(\frac{\partial\Phi}{\partial\mu}\right)^2 + \left(\frac{\partial\Phi}{\partial v}\right)^2\right) + \frac{\cosh^2\mu - \sin^2 v}{\cosh\mu\sin v}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\right)$$
$$-R^3\,\rho\,\Phi\,(\cosh^2\mu - \sin^2 v)\cosh\mu\sin v.$$

This yields for the equation of electrostatics

$$R\,\varepsilon_0 \frac{\partial}{\partial \mu}\left(\cosh\mu \sin\nu\ \frac{\partial \Phi}{\partial \mu}\right) + R\,\varepsilon_0 \frac{\partial}{\partial \nu}\left(\cosh\mu \sin\nu\ \frac{\partial \Phi}{\partial \nu}\right) + R\,\varepsilon_0 \frac{\partial}{\partial \varphi}\left(\frac{\cosh^2\mu - \sin^2\nu}{\cosh\mu \sin\nu}\ \frac{\partial \Phi}{\partial \varphi}\right)$$
$$+ R^3 \rho\,(\cosh^2\mu\ -\ \sin^2\nu)\cosh\mu \sin\nu\ =\ 0.$$

Finally then for the Laplacian and divergence:

$$\nabla^2 \Phi = \frac{1}{R^2(\cosh^2\mu - \sin^2\nu)\cosh\mu}\ \frac{\partial}{\partial \mu}\left(\cosh\mu\ \frac{\partial \Phi}{\partial \mu}\right) + \frac{1}{R^2(\cosh^2\mu - \sin^2\nu)\sin\nu}\ \frac{\partial}{\partial \nu}\left(\sin\nu\ \frac{\partial \Phi}{\partial \nu}\right) + \frac{1}{R^2\cosh^2\mu \sin^2\nu}\ \frac{\partial^2 \Phi}{\partial \varphi^2}$$

and, remembering

$$\vec{\nabla}\Phi = \frac{1}{R\sqrt{\cosh^2\mu - \sin^2\nu}}\ \frac{\partial \Phi}{\partial \mu}\,\vec{e}_\mu + \frac{1}{R\sqrt{\cosh^2\mu - \sin^2\nu}}\ \frac{\partial \Phi}{\partial \nu}\,\vec{e}_\nu + \frac{1}{R\cosh\mu \sin\nu}\ \frac{\partial \Phi}{\partial \varphi}\,\vec{e}_\varphi,$$

we get

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{R(\cosh^2\mu - \sin^2\nu)\cosh\mu}\ \frac{\partial}{\partial \mu}\left(\cosh\mu\ \sqrt{\cosh^2\mu\ -\ \sin^2\nu}\ E_\mu\right) +$$
$$\frac{1}{R(\cosh^2\mu - \sin^2\nu)\sin\nu}\ \frac{\partial}{\partial \nu}\left(\sin\nu\ \sqrt{\cosh^2\mu\ -\ \sin^2\nu}\ E_\nu\right) + \frac{1}{R\ \cosh\mu\ \sin\nu}\ \frac{\partial E_\varphi}{\partial \varphi}$$

The expression for the Laplacian looks pretty formidable, but in the case that $\rho = 0$ and there is circular symmetry around the z axis (so no φ dependence) the equation for the potential simplifies considerable. In this case,

$$\frac{1}{\cosh\mu}\ \frac{\partial}{\partial \mu}\left(\cosh\mu\ \frac{\partial \Phi}{\partial \mu}\right) + \frac{1}{\sin\nu}\ \frac{\partial}{\partial \nu}\left(\sin\nu\ \frac{\partial \Phi}{\partial \nu}\right) = 0,$$

which is much nicer.

11. Approximate Electrostatic Solutions by the Relaxation Method

The principle of minimizing (or, more generally, finding a stationary point for) the action to find the electrostatic potential in a specified 3D volume V with specified boundary potentials on ∂V can be used in practice to find an approximate solution to the problem. The idea is to represent the potential in some way with a function that contains adjustable parameters, for any values of which the boundary conditions are satisfied. Then calculate the action as a function of the parameters, and finally choose that set of parameters that makes the action stationary; this will be the best possible approximation of the given form to the actual potential.

The extreme case of this strategy is to parametrize the potential with itself, i.e., with its values on the vertices of a rectangular grid, defined by a (small) regular spacing Δ in x, y, and z. Deform the boundary slightly, as necessary, to follow the 3D grid. The vertices can be parametrized by integers $(i,\ j,\ k)$ so that the parameters are just $\Phi_{i,j,k}$ when $(i,\ j,\ k)$ does **not** identify a point on the boundary. Then approximate $\left(\frac{\partial \Phi}{\partial x}\right)_{i.j.k} \simeq \frac{\Phi_{i+1,j,k} - \Phi_{i,j,k}}{\Delta}$ and similarly for the other two dimensions. Also take $\rho_{i,j,k}$ as the given charge density in V at the vertex $(i,\ j,\ k)$. Finally, approximate the action integral by a sum and find the equations in the values of the potential that make it stationary. These equations can be solved iteratively by starting with, for example, zero potential on all the grid points inside V and the appropriate specified value on grid points on ∂V. This allows the calculation of a next approximation to the field. Then use it in the same equations, etc., and continue iterating until exhaustion calls a halt to the process (or more formally, the potential values become stable in value. In fact this procedure leads to a matrix equation and the problem becomes that of inverting a large sparse matrix. The solution by iteration can be shown to converge to the solution of the linear equations. This may seem pretty crude, but in fact it is quite practical and, with some embellishments, yields quite accurate solutions.

■ Solution

Begin by setting up a simple lattice of points in the volume V, $\vec{r}_{i,j,k}$ where $(i,\ j,\ k)$ are integers and

$$\vec{r}_{i+1,j,k} - \vec{r}_{i,j,k} = \Delta\,\vec{e}_x$$
$$\vec{r}_{i,j+1,k} - \vec{r}_{i,j,k} = \Delta\,\vec{e}_y$$
$$\vec{r}_{i,j,k+1} - \vec{r}_{i,j,k} = \Delta\,\vec{e}_z.$$

Let $\Phi_{i,j,k} = \Phi(\vec{r}_{i,j,k})$ and $\rho_{i,j,k} = \rho(\vec{r}_{i,j,k})$.

In general, the lattice points will not fall on the surface ∂V, but in general, the any point on the surface will be on order of Δ from a lattice point. So deform ∂V slightly (assume Δ is small) so that the deformed boundary $\partial V'$ passes through lattice points. At each of these assign the value of the potential to that on a nearby place in ∂V.

Next, approximate $\frac{\partial \Phi(\vec{r}_{i,j,k})}{\partial x} = \frac{\Phi_{i+1,j,k} - \Phi_{i,j,k}}{\Delta}$ and similarly for the other dimensions and finally approximate the action integral by a sum. Thus we have the approximation

$$\mathcal{A} \simeq \sum_{i,j,k} \left(\frac{\varepsilon_0}{2} \left(\left(\frac{\Phi_{i+1,j,k} - \Phi_{i,j,k}}{\Delta} \right)^2 + \left(\frac{\Phi_{i,j+1,k} - \Phi_{i,j,k}}{\Delta} \right)^2 + \left(\frac{\Phi_{i,j,k+1} - \Phi_{i,j,k}}{\Delta} \right)^2 \right) - \rho_{i,j,k}\, \Phi_{i,j,k} \right) \Delta^3 .$$

The values of the potential at all of the interior points of V are to be thought of as parameters to be chosen to make the action stationary under variation of these parameters, i.e., the derivative of the action with respect to each of these parameters must vanish. Thus if $(\ell,\ m,\ n)$ defines a point in the interior we get the equation

$$\begin{aligned} \frac{\varepsilon_0}{\Delta^2}\, \Delta^3\, (\Phi_{\ell,m,n} \,-\, \Phi_{\ell-1,m,n} - \Phi_{\ell+1,m,n} \,+\, \Phi_{\ell,m,n} \\ +\, \Phi_{\ell,m,n} \,-\, \Phi_{\ell,m-1,n} - \Phi_{\ell,m+1,n} \,+\, \Phi_{\ell,m,n} \\ +\, \Phi_{\ell,m,n} \,-\, \Phi_{\ell,m,n-1} - \Phi_{\ell,m,n+1} \,+\, \Phi_{\ell,m,n}) \\ -\rho_{\ell,m,n}\, \Delta^3 = 0. \end{aligned}$$

Rearranging, we get a form easy to visualize:

$$\begin{aligned} \Phi_{\ell,m,n} = \frac{1}{6}\, (\, \Phi_{\ell-1,m,n} + \Phi_{\ell+1,m,n} \\ +\, \Phi_{\ell,m-1,n} + \Phi_{\ell,m+1,n} \\ +\, \Phi_{\ell,m,n-1} + \Phi_{\ell,m,n+1}\,) + \frac{\Delta^2}{6\,\varepsilon_0}\, \rho_{\ell,m,n} \end{aligned}$$

which just says the potential at a lattice point is the average of the potentials on its six nearest lattice points, plus a contribution from any charge density there. If there is no dependence along one of the axes, say the z axis because of translational symmetry along it, we have a two dimensional problem and it is easy to see that the relaxation equation becomes

$$\Phi_{\ell,m,n} = \frac{1}{6}\,(\Phi_{\ell-1,m,n} + \Phi_{\ell+1,m,n} + \Phi_{\ell,m-1,n} + \Phi_{\ell,m+1,n} + \Phi_{\ell,m,n-1} + \Phi_{\ell,m,n+1}) + \frac{\Delta^2}{6\,\varepsilon_0}\,\rho_{\ell,m,n} \quad (3\,D,\ \rho \text{ is charge per unit volume})$$

$$\Phi_{\ell,m,n} = \tfrac{1}{4}\,(\Phi_{m-1,n} + \Phi_{m+1,n} + \Phi_{m,n-1} + \Phi_{m,n+1}) + \tfrac{\Delta^2}{4\,\varepsilon_0}\,\lambda_{m,n} \qquad (2\,D,\ \lambda \text{ is charge per unit area per unit length in the } z \text{ direction})$$

Example 1 in 2D

Suppose the 2D volume and boundaries are as indicated, with no charge in the volume:

$$\Phi \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \square & \square & \square & \square & 1 \\ 2 & \square & \square & \square & \square & 2 \\ 3 & \square & \square & \square & \square & 3 \\ 4 & \square & \square & \square & \square & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix}$$

Take the 0^{th} iteration to be zero potential on each of the interior points and show the first few iterations. Here is a little function to do the arithmetic.

```
In[85]:= relax[times_] :=
          Module[{Φ = {{0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 1}, {2, 0, 0, 0, 0, 2},
               {3, 0, 0, 0, 0, 3}, {4, 0, 0, 0, 0, 4}, {5, 5, 5, 5, 5, 5}}},
           Do[Φold = Φ; For[n = 2, n < 6, n++, For[m = 2, m < 6, m++, Φ[[n, m]] =
               1/4. (Φold[[n - 1, m]] + Φold[[n + 1, m]] + Φold[[n, m - 1]] + Φold[[n, m + 1]])]],
            {times}]; Print[MatrixForm@Φ]]
         relax[
          100]
```

The following shows the first few iterations, showing how the boundary propagate into the interior as the iterations proceed. Finally, since this volume has just constant electric fields on the "side" walls and equipotentials on the bottom and top walls, the result should be a uniform electric field inside the volume – as many iterations show is the case. The uniform potential gradient on the side walls can be created experimentally by running a uniform current through a uniform resistor making up the side walls. This idea is used in high voltage devices when a constant field is needed in some portion of space in an apparatus.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 & 0 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \underset{1}{\to} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & .25 & 0 & 0 & .25 & 1 \\ 2 & .5 & 0 & 0 & .5 & 2 \\ 3 & .75 & 0 & 0 & .75 & 3 \\ 4 & 2.25 & 1.25 & 1.25 & 2.25 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \underset{2}{\to}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.375 & 0.0625 & 0.0625 & 0.375 & 1 \\ 2 & .75 & 0.125 & 0.125 & .75 & 2 \\ 3 & 1.4375 & .5 & .5 & 1.4375 & 3 \\ 4 & 2.75 & 2.125 & 2.125 & 2.75 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \underset{3}{\to}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.453125 & 0.140625 & 0.140625 & 0.453125 & 1 \\ 2 & 0.984375 & 0.359375 & 0.359375 & 0.984375 & 2 \\ 3 & 1.75 & 1.04688 & 1.04688 & 1.75 & 3 \\ 4 & 3.14063 & 2.59375 & 2.59375 & 3.14063 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \to$$

$$\ldots \underset{30}{\to} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.997732 & 0.99633 & 0.99633 & 0.997732 & 1 \\ 2 & 1.99633 & 1.99406 & 1.99406 & 1.99633 & 2 \\ 3 & 2.99633 & 2.99406 & 2.99406 & 2.99633 & 3 \\ 4 & 3.99773 & 3.99633 & 3.99633 & 3.99773 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \to \ldots \underset{100}{\to} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1. & 1. & 1. & 1. & 1 \\ 2 & 2. & 2. & 2. & 2. & 2 \\ 3 & 3. & 3. & 3. & 3. & 3 \\ 4 & 4. & 4. & 4. & 4. & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix}$$

Example 2 in 2D

Again restrict to 2D, but this time put a line charge of ε_0 (taken as a pure number) Coulombs per meter (a lattice-sized 2D point charge} at the origin with a grounded circular boundary around it. Let the lattice spacing be Δ meters. Then the approximation for the charge density is that it have a value of $\frac{\varepsilon_0}{\Delta^2}$ in a single lattice cell. Thus the lattice size drops out of the problem and need not be specified.

In[11]:=
```
relaxch[times_] := Module[{mxx = 11}, mid = IntegerPart[mxx / 2.] + 1;
  Φ = (0 & /@ Range[mxx]) & /@ Range[mxx];
  ρ = Φ; ρ[[mid, mid]] = 1;
  Do[Φold = Φ; For[n = 2, n < mxx, n++,
    For[m = 2, m < mxx, m++, If[(n - mid)^2 + (m - mid)^2 < (mid - 1)^2,
      Φ[[n, m]] = 1/4. (Φold[[n - 1, m]] + Φold[[n + 1, m]] + Φold[[n, m - 1]] +
          Φold[[n, m + 1]] + ρ[[n, m]])]]], {times}]; Print[MatrixForm@Φ]]
relaxch[
 500]
```

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.25	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\underset{2}{\rightarrow}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.0625	0	0	0	0	0
0	0	0	0	0.0625	0.25	0.0625	0	0	0	0
0	0	0	0	0	0.0625	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\underset{3}{\rightarrow}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.03125 & 0.0625 & 0.03125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.015625 & 0.0625 & 0.3125 & 0.0625 & 0.015625 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.03125 & 0.0625 & 0.03125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \ldots \underset{100}{\to}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0.0220201 & 0.0340278 & 0.0383803 & 0.0340278 & 0.0 \\ 0 & 0 & 0.0270372 & 0.054108 & 0.0757108 & 0.0855564 & 0.0757108 & 0.0 \\ 0 & 0.0220201 & 0.054108 & 0.0916639 & 0.129316 & 0.152424 & 0.129316 & 0.0 \\ 0 & 0.0340278 & 0.0757108 & 0.129316 & 0.197466 & 0.265759 & 0.197466 & 0.1 \\ 0 & 0.0383803 & 0.0855564 & 0.152424 & 0.265759 & 0.515679 & 0.265759 & 0.1 \\ 0 & 0.0340278 & 0.0757108 & 0.129316 & 0.197466 & 0.265759 & 0.197466 & 0.1 \\ 0 & 0.0220201 & 0.054108 & 0.0916639 & 0.129316 & 0.152424 & 0.129316 & 0.0 \\ 0 & 0 & 0.0270372 & 0.054108 & 0.0757108 & 0.0855564 & 0.0757108 & 0.0 \\ 0 & 0 & 0 & 0.0220201 & 0.0340278 & 0.0383803 & 0.0340278 & 0.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0.0221497 & 0.0342099 & 0.0385925 & 0.0342099 & 0. \\ 0 & 0 & 0.0271945 & 0.054389 & 0.0760972 & 0.0859503 & 0.0760972 & 0. \\ 0 & 0.0221497 & 0.054389 & 0.0921144 & 0.12984 & 0.153014 & 0.12984 & 0. \\ 0 & 0.0342099 & 0.0760972 & 0.12984 & 0.198133 & 0.266427 & 0.198133 & 0 \\ 0 & 0.0385925 & 0.0859503 & 0.153014 & 0.266427 & 0.516427 & 0.266427 & 0. \\ 0 & 0.0342099 & 0.0760972 & 0.12984 & 0.198133 & 0.266427 & 0.198133 & 0 \\ 0 & 0.0221497 & 0.054389 & 0.0921144 & 0.12984 & 0.153014 & 0.12984 & 0. \\ 0 & 0 & 0.0271945 & 0.054389 & 0.0760972 & 0.0859503 & 0.0760972 & 0. \\ 0 & 0 & 0 & 0.0221497 & 0.0342099 & 0.0385925 & 0.0342099 & 0. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

You can clearly see the effect of the charge spreading out to the walls as the iterations grow. Further, there is clear circular symmetry. Finally since there is little change between 100 iterations and 500, stop at 500.

It is interesting to solve the analytic problem of the potential around a line charge centered in a grounded conducting cylinder of radius R and compare with this numerical solution.

Getting the free space 2D Green function is usually done with Gauss's law, but let's use the field equation for 2D electrostatics as n example of solving electrostatics problems from the differential equations. This problem is just the case of cylindrical geometry given above but with no z or φ dependence. Calling the potential $\Phi_2 = \Phi_2(\rho)$, we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_2}{\partial\rho}\right) = -\frac{\rho_{\text{charge}}}{\varepsilon_0}$$

where I will choose ρ_{charge} so that a spatial integral over a unit length in z and any area in the $x-y$ plane containing the origin yields ε_0. The expression surely must contain a factor $\delta(\rho)$ to express the concentration of charge along the z axis. But is that all? Let $\rho_{\text{charge}} = \varepsilon_0\, A\,\delta(\rho)$ where the prefactor A is to be chosen so that

$\varepsilon_0 = \int_{z=0}^{1}\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{\rho>0}\rho_{\text{charge}}\,\rho\, d\rho\, d\varphi\, dz = \int_{z=0}^{1}\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{\rho>0}\varepsilon_0\, A\,\delta(\rho)\,\rho\, d\rho\, d\varphi\, dz = 2\pi\varepsilon_0\int_{\rho=0}^{\rho>0} A\,\delta(\rho)\,\rho\, d\rho$ and so we must choose $A = \frac{1}{2\pi\rho}$ and interpret $\int_{\rho=0}^{\rho>0}\delta(\rho)\, d\rho$ as 1. Thus the equation to be solved is

$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_2}{\partial\rho}\right) = -\frac{1}{2\pi\rho}\delta(\rho) \;\;\Rightarrow\;\; \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_2}{\partial\rho}\right) = -\frac{1}{2\pi}\delta(\rho)$ (not worrying about multiplying by zero when ρ=0!).

Clearly if we choose $\Phi_2 = K\log\frac{R}{\rho}$, where K is any constant, we solve Poisson's equation and the boundary condition for $\rho \neq 0$. Next we integrate over ρ from 0 to any positive value to get

$$\int_0^{\rho>0}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_2}{\partial\rho}\right)d\rho = \rho\frac{\partial\Phi_2}{\partial\rho}\Big|_{\rho=0}^{\rho>0} = -\frac{1}{2\pi}\int_0^{\rho>0}\delta(\rho)\, d\rho = -\frac{1}{2\pi}.$$

It is not obvious how to handle the evaluation in the integrated part at $\rho = 0$. One way to handle the value of $\rho\frac{\partial\Phi_2}{\partial\rho}$ at $\rho = 0$ is to calculate it for a finite diameter cylinder of radius ϵ filled with a constant charge density $\frac{\varepsilon_0}{\pi\epsilon^2}$ per unit length centered in a grounded cylinder (just replace the delta function with a simple cylinder of charge) and then take the limit as ϵ goes to 0^+, i.e., $\lim_{\epsilon\to 0,\ \epsilon>0}$. It is easy to solve the problem

$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Phi_{\text{rod}}}{d\rho}\right) = \begin{pmatrix} -\frac{1}{\pi\epsilon^2} & 0\le\rho<\epsilon \\ 0 & \epsilon\le\rho\le R\end{pmatrix}$ giving $\rho\frac{d\Phi_{\text{rod}}}{d\rho} =$
$\begin{pmatrix} -\frac{\rho^2}{2\pi\epsilon^2}+K_1 & 0\le\rho<\epsilon \\ -K_2 & \epsilon\le\rho\le R\end{pmatrix}$ where K_1 and K_2 are constants. This function must be continuous since otherwise its derivative at the discontinuity would generate a delta function and Poisson's equation would be violated. Thus we must have $\frac{-1}{2\pi}+K_1 = -K_2$ or $\rho\frac{d\Phi_{\text{rod}}}{d\rho} = \begin{pmatrix} -\frac{\rho^2}{2\pi\epsilon^2}-K_2+\frac{1}{2\pi} & 0\le\rho<\epsilon \\ -K_2 & \epsilon\le\rho\le R\end{pmatrix}$. Integrating again gives

$\Phi_{\text{rod}} = \begin{pmatrix} -\frac{\rho^2}{4\pi\epsilon^2}+(-K_2+\frac{1}{2\pi})\log\rho\;+\;K_3 & 0\le\rho<\epsilon \\ K_2\log\frac{R}{\rho} & \epsilon\le\rho\le R\end{pmatrix}$ where K_2 and K_3 are constants and the boundary condition at $\rho = R$ has been satisfied.

As above impose the condition that Φ_{rod} must be continuous (so it does not have a delta function in its derivative which would violate Poisson's equation) and that it be regular everywhere. This then gives the conditions $K_2 = \frac{1}{2\pi}$ and $K_3 = K_2\log\frac{R}{\epsilon}+\frac{1}{4\pi}$. We get

$$\Phi_{\text{rod}} = \begin{pmatrix} -\frac{\rho^2-\epsilon^2}{2\pi\epsilon^2}+\frac{1}{2\pi}\log\frac{R}{\epsilon} & 0\le\rho<\epsilon \\ \frac{1}{2\pi}\log\frac{R}{\rho} & \epsilon\le\rho\le R\end{pmatrix}.$$

From this get $\rho\frac{\partial\Phi_2}{\partial\rho} = \begin{pmatrix} -\frac{\rho^2}{\pi\epsilon^2} & 0\le\rho<\epsilon \\ -\frac{1}{2\pi} & \epsilon\le\rho\le R\end{pmatrix}$, so we get the result that $\lim_{\epsilon\to 0,\,\epsilon>0} \rho \frac{\partial \Phi_2}{\partial \rho}(\rho = 0) = 0$. Finally then we get from

$$\int_0^{\rho>0} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi_2}{\partial \rho}\right) d\rho = \rho \frac{\partial \Phi_2}{\partial \rho}\Bigg|_{\rho=0}^{\rho>0} = \rho \frac{\partial \Phi_2}{\partial \rho}(\rho > 0) = K = -\frac{1}{2\pi}$$

and

$$\Phi_2 = \frac{1}{2\pi} \log \frac{R}{\rho}.$$

which is the usual result obtained from applying Gauss's law in three dimensions.

For our present purposes all we need is that $\Phi_2 = -\frac{1}{2\pi}\log\rho + \text{cnst}$. I will choose the constant so that the potential is zero at $\rho = 5$, approximating the square geometry above. Thus we get $\Phi_2(\rho) = \frac{1}{2\pi}\log\frac{5}{\rho}$ and

```
In[13]:= 1/(2 π) Log[5/#] & /@ {1., 2., 3., 4., 5.}

Out[13]= {0.25615, 0.145832, 0.0813004, 0.0355144, 0.}
```

Compare this to the values on the either the x or y axis from the relaxation approximation above,

```
{ 0.266427  0.153014  0.0859503  0.0385925  0 }
```

On a diagonal through the origin we have from the analytic formula

```
In[14]:= 1/(2 π) Log[5/(√2 #)] & /@ {1., 2., 3.}

Out[14]= {0.200991, 0.0906733, 0.0261415}
```

and from the relaxation with an approximate circular boundary,

```
{0.198133, 0.0921144, 0.0270372}
```

It seems reasonable that using a finer lattice should show that the approximation is in good agreement with the analytic solution as even this crude one yields a pretty good one.