# Ph196bEM Homework 3 Solutions - 2004

## 12. Jackson3ed, problem 1.7

Two long, cylindrical conductors of radii $a_1$ and $a_2$ are parallel and separated by a distance $d$ which is large compared to either radius. Show that the capacitance per unit length is given approximately by

$C = \frac{\pi \varepsilon_0}{\log(d/a)}$

where $a$ is the geometric mean of the two radii.

Approximately what gauge wire (give diameter in millimeters) would be necesssary to make a two-wire transmis
siln line with a capacitance of 1.2 $10^{-11}$ F/m if the separation of the wire is 0.5 cm? 1.5 cm? 5.0 cm?

### Solution

**start graphics**

**end graphics**

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Show[parallelwires, AspectRatio → .8 / 2.6]
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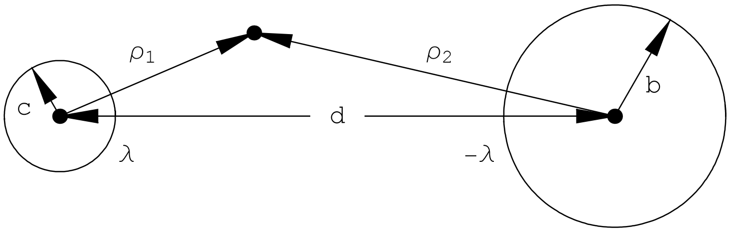

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Let the charge per unit length on the left wire be $\lambda$ and on the right, $-\lambda$. In the approximation that $d >> c$ and $d >> b$, we can simply assume the charge densities are uniformly distributed on the wire so the electric field due to the left wire is cylindrically symmetric around it and radially directed. From Gauss' law its magnitude is $\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho_1}$. The corresponding potential is $-\frac{\lambda}{2\pi\varepsilon_0}\ln(\rho_1)$. Similarly, the potential due to the wire on the right is $\frac{\lambda}{2\pi\varepsilon_0}\ln(\rho_2)$ and so the total potential at an arbitrary point is approximately $\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right)$.

Now use this to estimate the potential of each of the two wires. For a point on the surface of the one on the left we have $\rho_1 = c$ and $\rho_2 \simeq d$ so its potential is approximately $\frac{\lambda}{2\pi\varepsilon_0}\ln(\frac{d}{c})$. For the wire on the right, $\rho_1 \simeq d$ and

$\rho_2 = b$ so its potential is $\frac{\lambda}{2\pi\varepsilon_0}\ln(\frac{b}{d})$. Thus the potential drop from the left wire to the right one is $\frac{\lambda}{2\pi\varepsilon_0}(\ln(\frac{d}{c}) - \ln(\frac{b}{d})) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d^2}{c\,b}\right)$. Finally then the approximate capacitance per unit length is

$$\frac{\lambda}{\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d^2}{c\,b}\right)} = \frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$

where $a = \sqrt{b\,c}$ is the geometric mean of the two wire radii. So if the two wires have the same radius, it is just $a$. If the capacitance per unit length is $C$, then $a = d\,e^{-\pi\varepsilon_0/C}$. Estimate $a$ for certain parametric values.

```
wireDiameterInMM =
  1000 (2 # ⅇ^(-π ε0/C) /. {ε0 → 1/(4 π 10^-7 9 10^16), C → 1.2 10^-11}) & /@ {.005, .015, .05}
```

```
{0.987845, 2.96353, 9.87845}
```

The parameter is the spacing of the wires in meters.

## 13. Jackson3ed, problem 1.17.

A volume $V$ in vacuum is bounded by a surface $S$ consisting of several separate conducting surfaces $S_i$. One conductor is held at *unit* potential and all the others are at zero potential.

**(a)** Show that the capatance of the one conductor is given by

$$C = \varepsilon_0 \int_V \left|\vec{\nabla}\Phi\right|^2 d^3x$$

where $\Phi(\vec{r})$ is the solution for the potential.

**(b)** Show that the true capacitance $C$ is always less than or equal to the quantity

$$C[\Psi] = \varepsilon_0 \int_V \left|\vec{\nabla}\Psi\right|^2 d^3x$$

where $\Psi$ is any trial function satisfying the boundary conditions on the conductors. This is a variational principle for the capacitance that yields an *upper bound.*

### ■ Solution

**a)** The stored energy in the system is $U = \frac{1}{2}Q\,V = \frac{1}{2}C\,V^2 = \frac{\varepsilon_0}{2}\int_V E^2\,d^3r$ . For unit potential then, we have $C = \varepsilon_0 \int_V \left|\vec{\nabla}\Phi\right|^2 d^3r$.

**b)** Let $\psi = \Phi + \delta$ where $\Phi$ is the solution of the electrostatics problem and so satisfies the boundary conditions. Since by assumption $\psi$ also satisfies the boundary conditions, we get $\delta = 0$ on $\partial V$, the boundary of $V$. Consider the functional:

$$C[\psi] = \varepsilon_0 \int_V \left|\vec{\nabla}\psi\right|^2 d^3r = \varepsilon_0 \int_V \left|\vec{\nabla}\Phi + \vec{\nabla}\delta\right|^2 d^3r = \varepsilon_0 \int_V \left(\left|\vec{\nabla}\Phi\right|^2 + 2\,\vec{\nabla}\Phi\cdot\vec{\nabla}\delta + \left|\vec{\nabla}\delta\right|^2\right) d^3r.$$

Partially integrating the cross term we use $\vec{\nabla}\Phi\cdot\vec{\nabla}\delta = \vec{\nabla}\cdot\left(\delta\,\vec{\nabla}\Phi\right) - \delta\,\nabla^2\Phi = \vec{\nabla}\cdot\left(\delta\,\vec{\nabla}\Phi\right)$ because Laplace's equation is satisfied by $\Phi$ inside $V$. Applying Gauss' law and using $\delta = 0$ on $\partial V$ gives

$$C[\psi] = C + \varepsilon_0 \int_V \left|\vec{\nabla}\delta\right|^2 d^3r \geq C$$

with equality only when $\vec{\nabla}\delta = 0$ or $\psi =$ cnst. But since it is zero on $\partial V$, the cnst must be zero. Thus an estimate of the capacitance with this integral and any trial function will give a result larger than the actual capacitance.

## 14. Jackson3ed, problem 1.18

Consider the configuration of conductors of Problem 1.17, with all conductors except $S_1$ held at zero potential.

**(a)** Show that the potential $\boldsymbol{\Phi(\vec{x})}$ anywhere in the volume $V$ *and* on any of the surfaces $S_i$ can be written

$$\Phi(\vec{x}) = \int_{S_1} \sigma_1(\vec{x}')\, G(\vec{x}, \vec{x}')\, dA'$$

where $\sigma_1(\vec{x}')$ is the surface charge density on $S_1$ and $G(\vec{x}, \vec{x}')$ is the Green function potential for a unit point charge (a convention different from Jackson's) in the presence of all the surfaces that are held at zero potential (but with $S_1$ absent). Show also that the electrostatic energy is

$$W = \frac{1}{2} \int_{S_1} dA \int_{S_1} dA'\, \sigma_1(\vec{x})\, G(\vec{x}, \vec{x}')\, \sigma_1(\vec{x}')$$

where the integrals are only over the surface $S_1$.

**(b)** Show that the variational expression

$$C^{-1}[\sigma] = \frac{\int_{S_1} dA \int_{S_1} dA'\, \sigma(\vec{x})\, G(\vec{x},\vec{x}')\, \sigma(\vec{x}')}{\left(\int_{S_1} \sigma(\vec{x})\, dA\right)^2}$$

with an arbitrary integrable function $\sigma(\vec{x})$ defined on $S_1$, is stationary for small variations of $\sigma$ away from $\sigma_1$. Use Thompson's theorem to prove that the reciprocal of $C^{-1}[\sigma]$ gives a *lower bound* to the true capacitance of the conductor $S_1$.

### ■ Solution

**a)** Let the charge on boundary segment $S_1$ be $Q_1$, distributed as $\sigma_1(\vec{r} \in S_1)$, when this boundary segment is at the constant potential $\phi_1$ and all other parts of the surface are grounded. Clearly $Q_1 = \int_{S_1} \sigma_1(\vec{r})\, dA$.

In general, from Green's theorem, with $G$ defined with a unit point charge, we know that,

$$\Phi(\vec{r} \in V) = \int_V G(\vec{r}; \vec{r}')\, \rho(\vec{r}')\, d^3r' - \varepsilon_0 \int_{\partial V} \Phi(\vec{r}')\, \vec{\nabla}'\, G(\vec{r}; \vec{r}') \cdot d\vec{A}'.$$

(Note that when $G$ is defined with a unit charge, its SI dimensions are Volt/Coulomb, i.e., reciprocal capacitance and constants appear in different places than in Jackson's convention.)

For the case in this problem we choose $V$ to be a volume with surface $\partial V$ which includes all of the parts given in problem 1.17 *except for* $S_1$. The situation could look like this

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**End graphics**

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Show[boundary, AspectRatio → 1, ImageSize → 72 3]
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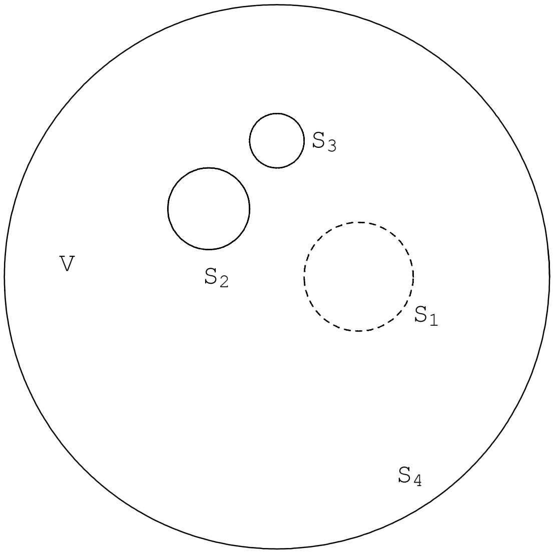

- Graphics -

Let $G(\vec{r};\vec{r}\,')$ be the Green function for the situation of a *unit* point charge at $\vec{r}\,'$ inside the volume $V$ **with the boundary segment $S_1$ removed.** Then let all of the other boundary segments be grounded and boundary segment $S_1$ covered with a surface charge density $\sigma_1(\vec{r}\in S_1)$ as described above. Then the second term in the general Green's theorem vanishes since $\Phi(\vec{r}\in \text{boundary of } V \text{ with } S_1 \text{ absent}) = 0$ and since the only charge density in the problem is that on the surface $S_1$. Thus we get

$$\Phi(\vec{r}\in V) = \int_{S_1} G(\vec{r};\vec{r}\,')\,\sigma_1(\vec{r}\,')\,dA'.$$

(Note that Jackson uses a different normalization for his Green function.) This relation is true all points in $V$ and by continuity, also on the surfaces $S_2,\ S_3,\ \ldots$ Furthermore for the given definition of $\sigma_1(\vec{r}\,')$ it is important to notice that

$$\Phi(\vec{r}\in S_1) = \int_{S_1} G(\vec{r}\in S_1;\vec{r}\,')\,\sigma_1(\vec{r}\,')\,dA' \ = \ \phi_1. \qquad \text{eq(1)}$$

For the energy, we use the general result that

$$W = \frac{1}{2}\int_V \Phi\,\rho\,d^3 r\ .$$

For the case at hand, the only charge is a surface charge on $S_1$ and so this becomes

$$W = \frac{1}{2} \int_{S_1} \Phi(\vec{r} \in S_1)\, \sigma_1 \, dA \; = \frac{1}{2} \int_{S_1} dA \int_{S_1} dA' \, \sigma_1(\vec{r}) \, G(\vec{r}; \vec{r}') \, \sigma_1(\vec{r}') \; .$$

From eq(1) it is trivial to see that this is just $\frac{1}{2} \phi_1 Q_1 = \frac{1}{2} \frac{Q_1^2}{C}$ where C is the capacitance of surface segment $S_1$ relative to all of the rest of the system at ground.

**b)** Consider the expression:

$$F[\sigma_1] = \frac{\int_{S_1} dA \int_{S_1} dA' \, \sigma_1(\vec{r}) \, G(\vec{r}; \vec{r}') \, \sigma_1(\vec{r}')}{\left(\int_{S_1} dA \, \sigma_1(\vec{r})\right)^2} \; .$$

Let $C$ be the capacitance of surface $S_1$ when all the rest of $\partial V$ is grounded. If the charge density $\sigma_1$is that which is on surface $S_1$ when it is an equipotential $\phi_1$ and the rest of $\partial V$ is grounded, then the numerator of this expression is twice the total energy stored in this electrostatic system, i.e., $C \phi_1{}^2 = \frac{Q_1{}^2}{C}$. and the denominator is just the square of the total charge on surface $S_1$, that is, $Q^2$. Thus, in this case, the expression is just the reciprocal of the capacitance $\frac{1}{C}$.

Now think of $F[\sigma(\vec{r})]$ as a functional and consider

$$\begin{aligned}
F[\sigma_1(\vec{r}) + \epsilon \; f(\vec{r})] &= \frac{\int_{S_1} dA \; \int_{S_1} dA' (\sigma_1(\vec{r}) + \epsilon \; f(\vec{r})) \, G(\vec{r}; \vec{r}') \, (\sigma_1(\vec{r}') + \epsilon \; f(\vec{r}'))}{\left(\int_{S_1} dA \, (\sigma_1(\vec{r}) + \epsilon \; f(\vec{r}))\right)^2} \\
&= \frac{1}{Q_1^2} \frac{1}{\left(1 + \epsilon \frac{1}{Q_1} \int_{S_1} dA \;\; f(\vec{r})\right)^2} \Big( \int_{S_1} dA \; \int_{S_1} dA' \, \sigma_1(\vec{r}) \, G(\vec{r}; \vec{r}') \, \sigma_1(\vec{r}') \\
&\qquad + \epsilon \int_{S_1} dA \; \int_{S_1} dA' (\sigma_1(\vec{r}) \; f(\vec{r}') + \sigma_1(\vec{r}') \; f(\vec{r})) \, G(\vec{r}; \vec{r}') \\
&\qquad + \epsilon^2 \int_{S_1} dA \; \int_{S_1} dA' \, f(\vec{r}) \, G(\vec{r}; \vec{r}') \; f(\vec{r}') \Big) \\
&= \frac{1}{Q_1^2} \frac{1}{\left(1 + \epsilon \frac{1}{Q_1} \int_{S_1} dA \;\; f(\vec{r})\right)^2} \Big( Q_1 \; \phi_1 \\
&\qquad + \epsilon \int_{S_1} dA \; \int_{S_1} dA' (\sigma_1(\vec{r}) \; f(\vec{r}') + \sigma_1(\vec{r}') \; f(\vec{r})) \, G(\vec{r}; \vec{r}') \\
&\qquad + \epsilon^2 \int_{S_1} dA \; \int_{S_1} dA' \, f(\vec{r}) \, G(\vec{r}; \vec{r}') \; f(\vec{r}') \Big)
\end{aligned}$$

where $f(\vec{r} \in S_1)$ is an arbitrary function defined on $S_1$ and $\epsilon$ is an infinitesimal. Expand in a series in $\epsilon$. The zeroth order term is just $F[\sigma_1] = \frac{\phi_1}{Q_1} = \frac{1}{C}$, and the first order term is

$$\epsilon \frac{1}{Q_1^2} \Big( \int_{S_1} dA \; \int_{S_1} dA' (\sigma_1(\vec{r}) \; f(\vec{r}') + \sigma_1(\vec{r}') \; f(\vec{r})) \, G(\vec{r}; \vec{r}') \; - Q_1 \; \phi_1 \, \frac{2}{Q_1} \int_{S_1} dA \;\; f(\vec{r}) \Big).$$

But using eq(1), you get for the first term (interchanging the order of integration and using $G(\vec{r}; \vec{r}') = G(\vec{r}\,'; \vec{r}\,)$:

$$\int_{S_1} dA' \;\; f(\vec{r}') \int_{S_1} dA \, \sigma_1(\vec{r}) \, G(\vec{r}' \in S_1; \vec{r}) \; = \; \phi_1 \int_{S_1} dA' \;\; f(\vec{r}').$$

Similarly for the second term:

$$\int_{S_1} dA \;\; f(\vec{r}') \int_{S_1} dA' \, \sigma_1(\vec{r}\,') \, G(\vec{r} \in S_1; \vec{r}\,') \; = \; \phi_1 \int_{S_1} dA \;\; f(\vec{r}).$$

So you get zero for the term of first order in $\epsilon$ and conclude that the functional $F[\sigma(\vec{r})]$ is stationary at $\sigma = \sigma_1$, the charge distribution that makes $S_1$ an equipotential, $\phi_1$. Notice that the functional does not depend upon the scale of $\sigma$ so that we could replace $\sigma$ everywhere by a normalized $\Sigma = Q\,\sigma / \int_{S_1} \sigma(\vec{r})\, dA$ and deal only with the constrained problem having $\int_{S_1} dA' \;\; f(\vec{r}') = 0$.

Finally, Thomson's theorem tells us that for specified charges on each of the boundary segments, the stored electrostatic energy is minimum when each of the boundary is a conductor, i.e., an equipotential. Thus by using normalized charge densities we conclude that $F[\sigma] \geq F[\sigma_1]$, or $1/F[\sigma] \leq C$, the capacity of the boundary segment $S_1$. This means that $1/F[\sigma]$ is a lower bound on $C$ for any trial surface charge density $\sigma$.

What makes this result less useful than it at first appears is that is assumes you can find the Green function, $G$, but, on the other hand, it is only needed on $S_1$ so such a restricted Green function, known only on this surface, is sufficient.

## 15. Jackson3ed, problem 1.20.

In estimating the capacitance of a given configuration of conductors, comparison with known capacitances is often helpful. Consider two configurations of $n$ conductors in which the $(1-n)$ conductors held at zero potential are the same, but the one conductor whose capacitance we wish to know is different. In particular, let the conductor in one configuration have a closed surface $S_1$ and in the other configuration have a surface $S_1'$, with $S_1'$ totally inside $S_1$

**(a)** Use the extremum principle of section 1.12 and the variational principle of Problem 1.17 to prove that the capacitance $C'$ of the conductor with the surface $S_1'$ is less than or equal to the capacitance $C$ of the conductor with the surface $S_1$ that encloses $S_1'$.

**(b)** Set upper and lower limits for the capacitance of a conducting cube of side $a$. Comparre your limits and also their average with the numerical value $C \simeq 0.655\,(4\,\pi\,\varepsilon_0\,a)$.

**(c)** By how much do you estimate the capacitance per unit length of the two-wire system of Problem 1.7 will change (larger? smaller?) if *one* of the wires is replaced by a wire with square cross section whose side is equal to the diameter of the other wire?

### ■ Solution

**a)** Let all surfaces except one ($S$ or $S'$) be at zero potential. Surface $S$ or $S'$ is at unit potential. Finally, $S$ is closed and $S'$ is everywhere inside it. Let $C$ be the capacitance of the situation when $S$ is the bounding surface, and $C'$ when $S'$ is. It is intuitively obvious that $C > C'$ since $S$ has more area then $S'$ and it is generally closer to the grounded conductors than is $S'$, both tending to increase the capacitance. This problem shows this intuitive result with a formal argument.

Let $\Phi(\vec{r})$ be the potential for the situation in which the surface $S$ is at unit potential. We can take $\vec{r}$ to be anywhere, including inside the volume enclosed by $S$. Of course if $\vec{r}$ is inside $S$, $\Phi(\vec{r})$ is unity, the potential of the surface $S$. From

$\frac{1}{2} Q V = \frac{1}{2} C V^2 = \frac{\varepsilon_0}{2} \int E^2 d^3 r,$

the capacitance of this situation is $C = \varepsilon_0 \int_{\vec{r} \text{ outside} S} \left|\vec{\nabla} \Phi\right|^2 d^3 r$.

Now use the same function $\Phi(\vec{r})$ as a trial function in calculating the capacitance when $S'$ is the bounding surface (at unit potential!). It satisfies all the boundary conditions so is an acceptable trial function. But, note that inside $S$, $\vec{\nabla}\Phi = 0$. From the fact that we get a number LARGER than the capacitance $C'$ when we use any function which is not the solution to Laplace's equation for the $S'$ situation, we can conclude that

$C' \leq \varepsilon_0 \int_{\vec{r} \text{ outside} S'} \left|\vec{\nabla} \Phi\right|^2 d^3 r$
$= \varepsilon_0 \int_{\vec{r} \text{ outside} S} \left|\vec{\nabla} \Phi\right|^2 d^3 r$
$= C.$

So you have the intuitive and general result : C' ≤ C.

**b)** The capacitance of a sphere of radius $r$ is just $C_{\text{sph}} = 4 \pi \varepsilon_0 r$. The capacitance of a cube with side $a$ is smaller than that of an enclosing sphere which has radius $\frac{\sqrt{3}\, a}{2}$. Thus $C_{\text{cube} a} < \frac{\sqrt{3}}{2} (4 \pi \varepsilon_0 a)$. Also, the radius of a sphere inscribed in a cube of side $a$ is just $a/2$. In this case the sphere has a smaller capacitance than the cube so we get $C_{\text{cube} a} > \frac{1}{2} (4 \pi \varepsilon_0 a)$. So $\frac{1}{2} (4 \pi \varepsilon_0) a < C_{\text{cube} a} < \frac{\sqrt{3}}{2} (4 \pi \varepsilon_0)$. Numerical approximation gives $C_{\text{cube} a} = .655 (4 \pi \varepsilon_0 a)$, and indeed $0.5 < .655 < .866$. The average of the two limits is 0.683 - not such a bad estimate!

```
√3 / 2.
0.866025
```

```
(1 / 2 + √3 / 2) / 2 // N
0.683013
```

**c)** Problem 1.7 gives the approximate capacitance between two long cylindrical conductors of radii $a_1$ and $a_2$ with centers separated by $d$ as $C = \frac{\pi \varepsilon_0}{\log\left(d/\sqrt{a_1 a_2}\right)}$. Suppose that $a_1 \leq a_2$.

Case1) Now let the cylinder of radius $a_1$ be replaced by a square conductor of side $2 a_2$. The situation is as in the following plot (solid black is the original, solid gray is the replacement, and the two dashed circles are for estimating the capacitance of the square.

**begin graphics**

**end graphics**

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Show[case1, AspectRatio -> 1, ImageSize → 2 72]
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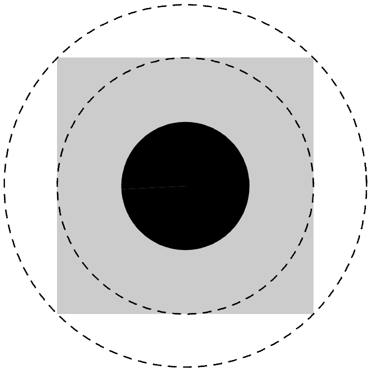

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Clearly the capacitance will increase when the square is the conductor since its surface is closer to the other conductor (the result of the formal theorem). To estimate the capacitance, average that of the capacitance with the inscribed circle of radius $a_2$ and circumscribed one of radius $\sqrt{2}\ a_2$. The result is

$$C_{\text{square}\,a_2\text{ replacing circle}\,a_1<a_2} \simeq \frac{\pi\varepsilon_0}{2}\left[\frac{1}{\log(d/a_2)} + \frac{1}{\log(d/a_2)-\frac{1}{4}\log 2}\right].$$

For example, if $a_1 = 1$ mm, $a_2 = 2$ mm and $d = 20$ mm, we get as the original capacitance in cm/cm ≃ pF/cm (formally replacing $\varepsilon_0$ by $1/4\pi$ gives the cgs formula, which I use below)

$$\texttt{cOrig} = \frac{1}{4\,\texttt{Log}\left[20\big/\sqrt{2}\right]}\ \texttt{// N}$$

```
0.0943696
```

$$\texttt{cReplace1Approx} = \frac{1}{8}\left(\frac{1}{\texttt{Log[ 20 / 2]}} + \frac{1}{\texttt{Log[ 20 / 2]} - \frac{1}{4}\,\texttt{Log [2]}}\right)\ \texttt{// N}$$

```
0.112992
```

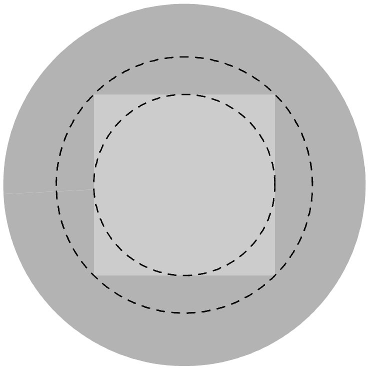
So the replacement makes the capacitance slightly larger, as it must.

Case 2) Now let the cylinder of radius $a_2$ be replaced by a square conductor of side $2a_1$. The situation is as in the following plot (original conductor in dark gray replaced by square in light gray. The two dashed circles are for estimating the capacitance of the square case.

**begin graphics**

**end graphics**

```
Show[case2, AspectRatio -> 1, ImageSize → 2 72]
```



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- Graphics -
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Clearly the capacitance will be reduced. An estimate of the final capacitance is

$$C_{\text{square }a_1\text{ replacing circle }a_2>a_1} \simeq \frac{\pi\varepsilon_0}{2}\left[\frac{1}{\log(d/a_1)}+\frac{1}{\log(d/a_1)-\frac{1}{4}\log 2}\right].$$

For the same numbers as above, get

$$\text{cOrig} = \frac{1}{4\,\text{Log}\left[20\big/\sqrt{2}\,\right]}\ //\ \text{N}$$

```
0.0943696
```

$$\text{cReplacea1Approx} = \frac{1}{8}\left(\frac{1}{\text{Log}[\,20/1]}+\frac{1}{\text{Log}[\,20/1]-\frac{1}{4}\,\text{Log}\,[2]}\right)\ //\ \text{N}$$

```
0.0860139
```

So the replacement makes the capacitance slightly smaller, as it must be.

## 16. A Capacitance Matrix

A large circuit board (30 cm by 30 cm) is covered with a thin layer of copper to act as a ground plane (i.e., the copper is connected to the laboratory's ground system and is at zero potential). For mechanical convenience, the copper is bonded to a thin insulating plastic board for support. Assume that no components are mounted on the circuit board so the ground plane is "clean". Near its center are three circular holes, each of diameter 2mm, fully etched through the copper. The centers of the three holes are on the vertices of an equilateral triangle with 0.5 cm sides. Centered in each of the holes is a circular copper pad of diameter 1 mm, well insulated from the copper ground plane.

Of course, it's too hard to analytically calculate electrostatics problems relating to the pads in this geometry (except numerically and that's a lot of work), but the only electrical information that will be needed in the final application of the circuit board are various capacitances. Unfortunately, someone has ripped off your handy capacitance meter, but, rummaging around the lab, you find a fancy electrostatic voltmeter, a 100 volt power supply, and a precision integrating ammeter, i.e., an instrument that can tell you the integral of the current that has flowed through it. Of course, you know that an electrostatic voltmeter measures the voltage between two terminals without drawing any current from them . Suppose you can assume that the voltmeter's capacitance to ground is negligible compared to any other capacitance in the problem. With the negative lead from the power supply always connected to ground, you do the following sequence of operations and measurements. Assume in all of the following that the wires used in hooking things up are thin enough and are laid out carefully enough to have negligible effects. You may find it useful to make a circuit theory sketch illustrating the steps taken.

i) First, the electrostatics system is reset, i.e., the three pads are connected together and then to ground. The wires are then removed.

ii) One of the pads is then reconnected to ground and one of the others is connected to the positive lead of the 100 volt power supply through the integrating ammeter. Care is taken to leave the third pad undisturbed. The integrator shows that a charge of 1 nanoCoulomb ($10^{-9}$ C) flows to the pad as it charges to 100 V above ground.

iii) The connections to the pads are removed, and, as at the beginning, the system is reset, and then all the wires are removed.

iv) Lastly, the positive lead of the 100 V power supply is connected to one of the pads. The voltage of one of the other two pads is then measured with the electrostatic voltmeter and found to be 25 V above ground.

**a)** From the given data write the capacitance matrix for the geometry in units of pF (picoFarads).

**b)** If you leave the power supply and voltmeter connected as in item iv) above, and then touch the power supply's positive lead to the third pad, what will the voltmeter read (give the numerical answer in volts)?

**c)** Finally, you carefully disconnect the power supply from the two pads and replace the voltmeter with a 10 Ohm resistor to ground. How much energy (give the result numerically in nJ, nanoJoules) is dissipated in the resistor by the current that flows through it?

Notice that the electrostatic field is negligible near the edges of the ground plane so you can consider them to be at infinity. Thus the geometry has three-fold rotational symmetry. Exploit it in writing the capacitance matrix.

## ■ Solution

a) From the symmetry, we can immediate relate the charges on each to the pads to the potentials on each with the capacitance matrix:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_1 & -c_2 & -c_2 \\ -c_2 & c_1 & -c_2 \\ -c_2 & -c_2 & c_1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \text{ or in matrix notation, } \underset{\sim}{q} = \underset{\approx}{C}\, \underset{\sim}{\Phi}\,.$$

In the capacitance matrix both $c_1$ and $c_2$ are both positive, non-zero numbers. This is easy to see intuitively by thinking of lines of force when you ground all but one of the pads and put a positive charge on the third. It can be gotten formally from the theorem that:

$\alpha$) the capacitance matrix is always positive definite and so has a positive, non-zero determinate and positive diagonal elements.

$\beta$) all of the matrix elements of its inverse are positive numbers.

i) After we reset the system $\underset{\sim}{q} = 0$ since resetting makes $\underset{\sim}{\Phi} = 0$.

ii) Letting $V_0 = 100$ Volts and $Q = 1$ nanoCoulomb, we get after step ii) (since one pad, I choose 2, is untouched it still has no charge but it floats to the appropriate electrostatic potential):

$$\begin{pmatrix} q_1 \\ 0 \\ Q \end{pmatrix} = \begin{pmatrix} c_1 & -c_2 & -c_2 \\ -c_2 & c_1 & -c_2 \\ -c_2 & -c_2 & c_1 \end{pmatrix} \begin{pmatrix} 0 \\ \varphi_2 \\ V_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 = c_1\,\varphi_2 - c_2\,V_0 \\ Q = -c_2\,\varphi_2 + c_1\,V_0 \end{pmatrix} \Rightarrow \varphi_2 = \frac{c_2}{c_1}\,V_0 \Rightarrow Q = \left(-\frac{c_2^2}{c_1} + c_1\right) V_0$$

iii) We now reset again so all the pads are all uncharged after this step..

iv) Let $V_1 = 25$ Volts. The actions now imply that

$$\begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_1 & -c_2 & -c_2 \\ -c_2 & c_1 & -c_2 \\ -c_2 & -c_2 & c_1 \end{pmatrix} \begin{pmatrix} V_1 \\ \varphi_2' \\ V_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 = c_1\,V_1 - c_2\,\varphi_2' - c_2\,V_0 \\ 0 = -c_2\,V_1 + c_1\,\varphi_2' - c_2\,V_0 \end{pmatrix}$$

Subtract these two equations to get the result (obvious from symmetry): $\varphi_2' = V_1$ and so from the first equation $0 = (c_1 - c_2)\,V_1 - c_2\,V_0 \Rightarrow c_2 = c_1\,\frac{V_1}{V_0+V_1} = \frac{1}{5}\,c_1$.

Then from step ii) you get $\frac{Q}{V_0} = c_1(1 - \frac{1}{25}) = \frac{10^{-9}}{10^2}$ Farads $= c_1\,\frac{24}{25} \;\Rightarrow c_1 = \frac{125}{12}$ pF. This then gives $c_2 = \frac{1}{5}\,\frac{125}{12}$ pF $= \frac{25}{12}$ pF. So

$$\underset{\approx}{C} = \frac{25}{12}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

**b)** Note that pad 1 is still charge free since nothing has touched it. So whe the other two pads are set to 100 Volts, we get

$$\begin{pmatrix} 0 \\ q_2 \\ q_3 \end{pmatrix} = \frac{25}{12}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ 100 \\ 100 \end{pmatrix} \Rightarrow\; 0 = 5\,\varphi_1 \;-200\,\text{Volts} \Rightarrow\; \varphi_1 = 40\,\text{Volts}.$$

So the the voltmeter will red 49 Volts.

**c)** The amout of energy dissipated in the resistor as one of the pads discharge through it is of course just the difference in the stored energy in the electrostatic field before and after the discharge. Before the discharge we have

$$\begin{pmatrix} 0 \\ q_2 \\ q_3 \end{pmatrix} = \frac{25}{12}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 40 \\ 100 \\ 100 \end{pmatrix} \;\Rightarrow\; q_2 = q_3 = \frac{25}{12}\,(-40 + 400)\,\text{pC} = 750\,\text{pC}$$

and the amount of energy in the field before the discharge is

$$W_{\text{before}} = \frac{1}{2}\,(q_1\,\Phi_1 + q_2\,\Phi_2 + q_3\,\Phi_3)_{\text{before}} = \frac{1}{2}\,(0\times 40 + 750\,\times 100 + 750\,\times 100)\,\text{pJ} = 75\,\text{nJ}.$$

The potential on the first pad goes to zero when the discharge is complete, but it will still have a charge on it – call it $q_1$ – and te other two pads will have potentials – call them $\varphi_2$ and $\varphi_3$ (the same by symmetry, of course). So we have after the discharge

$$\begin{pmatrix} q_1 \\ 750 \\ 750 \end{pmatrix} = \frac{25}{12}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 750 = \frac{25}{12}\,(5\,\varphi_2 - \varphi_3) \\ 750 = \frac{25}{12}\,(-\varphi_2 + 5\,\varphi_3) \end{pmatrix} \Rightarrow\; \varphi_2 = \varphi_3 \;= \frac{3\times 750}{25} \;= 90\,\text{Volts}.$$

Consequently the stored energy after the discharge is

$$W_{\text{after}} = \frac{1}{2}\,(q_1\,\Phi_1 + q_2\,\Phi_2 + q_3\,\Phi_3)_{\text{after}} = \frac{1}{2}\,(0\times q_1 + 750\,\times 90 + 750\,\times 90)\,\text{pJ} \;=\; 7.5\times 9\,\text{nJ}.$$

So finally, the energy dissipated into heat in the resistor is $(75 - 75\times 0.9) = 7.5$ nJ.

There are several other ways to do this problem, using circuit theory capacitors or symmetry arguments.