

# Ph196bEM Homework 4 Due: Mar. 13, 10:40am

## ■ NOTE: Since there is no class on Mar 13, this homework is to be turned in at the box in front of Lau 335

19. This problem is to find two different representation of the Dirichlet Green function  $G(x, y; x', y')$  inside a channel defined by the boundaries  $(x = -d/2, y \in [0, \infty))$ ,  $(x \in [-d/2, d/2], y = 0)$ , and  $(x = d/2, y \in [0, \infty))$ , where  $d > 0$ . As usual, let  $x_>$  ( $y_>$ ) be the greater of  $x$  ( $y$ ) and  $x_<$  ( $y_<$ ), the smaller.

It can be shown by using a method called conformal mapping (not a part of this course) that the Green function for this problem can be expressed in the following closed form:

$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon_0} \log \left( \frac{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) - \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2}{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) + \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2} \right).$$

The problem is not to show this. I give it so you can compare it to the following results, for example, numerically.

a) Splitting the space into two parts with a line through  $(x', y')$  parallel to the  $x$  axis, show that

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \sinh(n\pi \frac{y_<}{d}) e^{-n\pi \frac{y_>}{d}} \sin(n\pi (\frac{x}{d} + \frac{1}{2})) \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

b) Next show the completeness relation that, when  $k > 0$  and  $k' > 0$ ,

$$\int_0^{\infty} \sin(kx) \sin(k'x) dx = \frac{\pi}{2} \delta(k - k').$$

This follows readily from  $\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi \delta(k)$  with a little manipulation (but don't forget the conditions on  $k$  and  $k'$ ).

c) Finally give an integral representation for  $G$ ,

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \int_0^{\infty} \frac{1}{k \sinh(kd)} \sinh(k(d/2 + x_<)) \sinh(k(d/2 - x_>)) \sin(ky) \sin(ky') dk.$$

20. Find the Green function for the inside of a rectangular box with dimensions  $a$  in the  $x$ -direction,  $b$  in the  $y$ -direction, and  $c$  in the  $z$ -direction. For convenience, let one of the corners of the box be at the origin - use figure 2.9 of Jackson to define the coordinate system. Further, use real exponential functions in the  $z$  coordi-

nate in your answer. Of course, it is easy then to write other representations of the solution with the real exponentials in the  $x$  or  $y$  direction. **Do not use the method of images for this problem.**

21. Find the force acting in the  $z$  direction on a point charge of magnitude  $q$  located at point  $(x', y', z')$  inside the box given in problem 21 if all of its walls are at zero potential.

Hint: Be sure to realize that the Green function found in problem 21 is singular at the point

$\vec{r} = (x, y, z) = (x', y', z') = \vec{r}'$  and so the  $z$  force on  $q$  is *not*  $-q^2 \frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$  because  $G$  contains contributions from both the direct field of the point charge and that due to the induced charges in the walls of the box.

Either

a) don't forget that Newton's third law applies in electrostatics,

or

b) notice that the singularity in  $\frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$  can be removed by considering

$$\frac{1}{2} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \left( - \frac{\partial G(x', y', z' - \epsilon; x', y', z')}{\partial z} - \frac{\partial G(x', y', z' + \epsilon; x', y', z')}{\partial z} \right).$$

22. Jackson, problem 3.1. Don't forget about the three recursion relations for Legendre polynomials  $P_l(x)$  given in Jackson's formula 3.29.

23. Jackson, problem 3.2. One way to approach this problem is to use the short discussion related to Jackson formula 3.37. For working out the potential on the  $z$  axis, use the Legendre polynomial generating function, formula 3.38. Another approach is to note that the potential must be continuous across the sphere of radius  $R$  whereas the electric field will have a discontinuity across its charged part. Further the potential must go to zero at infinity and be finite at the origin.