

Ph196bEM Homework4 Due: Friday, Mar 12, 5pm, 337 Lauritsen

17. This problem is to find two different representation of the Dirichlet Green function $G(x, y; x', y')$ inside a channel defined by the boundaries $(x = -d/2, y \in [0, \infty))$, $(x \in [-d/2, d/2], y = 0)$, and $(x = d/2, y \in [0, \infty))$, where $d > 0$. As usual, let $x_>$ ($y_>$) be the greater of x (y) and x' (y'), and $x_<$ ($y_<$), the smaller.

It can be shown by using the method of conformal mapping (not a part of this course) that the Green function for this geometry can be expressed in the following closed form:

$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon_0} \log \left(\frac{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) - \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2}{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) + \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2} \right).$$

The problem is not to show this. I give it so you can compare it to the following results – numerically, for example.

a) Splitting the space into two parts with a line through (x', y') parallel to the x axis, show that

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \sinh(n\pi \frac{y_<}{d}) e^{-n\pi \frac{y_>}{d}} \sin(n\pi (\frac{x}{d} + \frac{1}{2})) \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

The " $\frac{1}{n}$ " as a factor in the series is characteristic of a logarithmic solution. Its continuum analog appears in the integral representation given next.

b) Next show the completeness relation that, when $k > 0$ and $k' > 0$,

$$\int_0^{\infty} \sin(kx) \sin(k'x) dx = \frac{\pi}{2} \delta(k - k').$$

This follows readily from $\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi \delta(k)$ with a little manipulation (but don't forget the conditions on k and k').

c) Finally show that an integral representation for G is,

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \int_0^{\infty} \frac{1}{k \sinh(kd)} \sinh(k(d/2 + x_<)) \sinh(k(d/2 - x_>)) \sin(ky) \sin(ky') dk.$$

18. Find the potential inside a long straight segment of a round pipe of radius a . Using two dimensional polar coordinates (ρ, φ) , let the volume be defined by $0 < \rho < a$, and $0 \leq \varphi \leq \varphi_0$ where $\varphi_0 > 0$. The boundary conditions are that $\Phi(0 < \rho \leq \rho_0, \varphi = 0) = \Phi(\rho = a, 0 \leq \varphi < \varphi_0) = 0$ and $\Phi(0 < \rho \leq \rho_0, \varphi = \varphi_0) = V$. Do not use the Green function constructed with oscillatory functions in the φ variable; I want you to use solutions with oscillatory solutions in the ρ direction.

19. Find the Green function for the inside of a rectangular box with dimensions a in the x -direction, b in the y -direction, and c in the z -direction. For convenience, let one of the corners of the box be at the origin - use figure 2.9 of Jackson to define the coordinate system. Further, use real exponential functions in the z coordinate

in your answer. Of course, it is easy then to write other representations of the solution with the real exponentials in the x or y direction. **Do not use the method of images for this problem.**

20. Find the force acting in the z direction on a point charge of magnitude q located at point (x', y', z') inside the box given in problem 19 if all of its walls are at zero potential.

Hint: Be sure to realize that the Green function found in problem 19 is singular at the point

$\vec{r} = (x, y, z) = (x', y', z') = \vec{r}'$ and so the z force on q is *not* $-q^2 \frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$ because G contains contributions from both the direct field of the point charge and that due to the induced charges in the walls of the box.

Either

a) don't forget that Newton's third law applies in electrostatics,

or

b) notice that the singularity in $\frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$ can be removed by considering

$$\frac{1}{2} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \left(-\frac{\partial G(x', y', z' - \epsilon; x', y', z')}{\partial z} - \frac{\partial G(x', y', z' + \epsilon; x', y', z')}{\partial z} \right).$$

21. Jackson, problem 2.13. You may find the relation $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan[x]$ useful.

22. Jackson, problem 3.10. Don't forget about the Bessel functions that are like exponentials with real argument. They are called the *modified* Bessel functions.