

Ph196bEM Homework 4 Solutions - 2004

■ Problem 17

This problem is to find two different representation of the Dirichlet Green function $G(x, y; x', y')$ inside a channel defined by the boundaries $(x = -d/2, y \in [0, \infty))$, $(x \in [-d/2, d/2], y = 0)$, and $(x = d/2, y \in [0, \infty))$, where $d > 0$. As usual, let $x_>$ ($y_>$) be the greater of x (y) and x' (y'), and $x_<$ ($y_<$), the smaller.

It can be shown by using the method of conformal mapping (not a part of this course) that the Green function for this geometry can be expressed in the following closed form:

$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon_0} \log \left(\frac{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) - \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2}{(\sin(\pi \frac{x}{d}) \cosh(\pi \frac{y}{d}) - \sin(\pi \frac{x'}{d}) \cosh(\pi \frac{y'}{d}))^2 + (\cos(\pi \frac{x}{d}) \sinh(\pi \frac{y}{d}) + \cos(\pi \frac{x'}{d}) \sinh(\pi \frac{y'}{d}))^2} \right).$$

The problem is not to show this. I give it so you can compare it to the following results – numerically, for example .

a) Splitting the space into two parts with a line through (x', y') parallel to the x axis, show that

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \sinh(n\pi \frac{y_<}{d}) e^{-n\pi \frac{y_>}{d}} \sin(n\pi (\frac{x}{d} + \frac{1}{2})) \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

The " $\frac{1}{n}$ " as a factor in the series is characteristic of a logarithmic solution. Its continuum analog appears in the integral representation given next.

b) Next show the completeness relation that, when $k > 0$ and $k' > 0$,

$$\int_0^{\infty} \sin(kx) \sin(k'x) dx = \frac{\pi}{2} \delta(k - k').$$

This follows readily from $\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi \delta(k)$ with a little manipulation (but don't forget the conditions on k and k').

c) Finally show that an integral representation for G is,

$$G(x, y; x', y') = \frac{2}{\pi\epsilon_0} \int_0^{\infty} \frac{1}{k \sinh(kd)} \sinh(k(d/2 + x_<)) \sinh(k(d/2 - x_>)) \sin(ky) \sin(ky') dk.$$

Solution Problem 17

a) Split the channel into the two regions $0 \leq y \leq y'$ and $y' \leq y$. Then in the x coordinate we use sinusoids which vanish at $x = \pm d/2$. So we have

$$G = \sum_{n=1}^{\infty} g_n(y) \sin(n\pi (\frac{x}{d} + \frac{1}{2}))$$

which vanishes at $x = \pm \frac{d}{2}$.

To have a solution of Laplace's equation everywhere except on the point charge we take

$$g_n(y) = A_n \begin{pmatrix} \sinh(n\pi \frac{y}{d}) e^{-n\pi \frac{y'}{d}}, & \text{for } 0 \leq y \leq y' \\ \sinh(n\pi \frac{y'}{d}) e^{-n\pi \frac{y}{d}}, & \text{for } y' \leq y \end{pmatrix}.$$

This function is arranged so that G satisfies Laplace's equation where necessary, it vanishes at $y = 0$ and as $y \rightarrow \infty$, it is continuous at $y = y'$ so that the Laplacian of G does not produce the derivative of a δ function, and, finally, it contains a scaling factor so that Poisson's equation for a unit charge can be satisfied.

Poisson's equation for this Green function is

$$\nabla^2 G = \sum_{n=1}^{\infty} (g_n'' - (\frac{n\pi}{d})^2 g_n) \sin(n\pi (\frac{x}{d} + \frac{1}{2})) = -\frac{1}{\epsilon_0} \delta(x - x') \delta(y - y').$$

The orthogonality relation for these sines is easily seen to be

$$\int_{-d/2}^{d/2} \sin(n\pi (\frac{x}{d} + \frac{1}{2})) \sin(m\pi (\frac{x}{d} + \frac{1}{2})) dx = \int_0^d \sin(n\pi \frac{\zeta}{d}) \sin(m\pi \frac{\zeta}{d}) d\zeta = \delta_{mn} \frac{d}{2}.$$

Use this to dissolve the sum into a single term and (renaming the dummy variable m to be n) get

$$\frac{d}{2} (g_n'' - (\frac{n\pi}{d})^2 g_n) = -\frac{1}{\epsilon_0} \delta(y - y') \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

Finally, integrate over the singularity in y from $y' - \epsilon$ to $y' + \epsilon$, $\epsilon > 0$, and take the limit as $\epsilon \rightarrow 0$. This gives

$$\frac{d}{2} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} (g_n'(y' + \epsilon) - g_n'(y' - \epsilon)) = -\frac{1}{\epsilon_0} \sin(n\pi (\frac{x'}{d} + \frac{1}{2})),$$

or

$$\frac{d}{2} \frac{n\pi}{d} A_n (-\sinh(n\pi \frac{y'}{d}) e^{-n\pi \frac{y'}{d}} - \cosh(n\pi \frac{y'}{d}) e^{-n\pi \frac{y'}{d}}) = -\frac{1}{\epsilon_0} \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

Finally then $A_n = \frac{2}{n\pi\epsilon_0} \sin(n\pi (\frac{x'}{d} + \frac{1}{2}))$ and the Green function is

$$G = \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \sinh(n\pi \frac{y_{<}}{d}) e^{-n\pi \frac{y_{>}}{d}} \sin(n\pi (\frac{x}{d} + \frac{1}{2})) \sin(n\pi (\frac{x'}{d} + \frac{1}{2})).$$

b) If we split the channel by a line parallel to the y axis, we need an integral over a continuum of k values rather than a discrete set of them. We will need the completeness for the sine Fourier integral, as it is called. For $k > 0$ and $k' > 0$, consider

$$\begin{aligned} \int_0^{\infty} \sin(kx) \sin(k'x) dx &= \frac{1}{2} \int_0^{\infty} (\cos((k-k')x) - \cos((k+k')x)) dx \\ &= \frac{1}{4} \int_{-\infty}^{\infty} (\cos((k-k')x) - \cos((k+k')x)) dx \\ &= \frac{1}{4} \operatorname{Re} \left(\int_{-\infty}^{\infty} (e^{i(k-k')x} - e^{i(k+k')x}) dx \right) \\ &= \frac{\pi}{2} (\delta(k-k') - \delta(k+k')). \end{aligned}$$

But the argument $k+k'$ has no zeros when $k > 0$ and $k' > 0$, and so we get

$$\int_0^{\infty} \sin(kx) \sin(k'x) dx = \frac{\pi}{2} \delta(k-k').$$

c) If we split the channel by cutting it along $x = x'$, then we use a sinusoid in the y coordinate and exponentials in x . There are not now two finite values of the argument of the sinusoid at which G is to vanish, so we must use a continuum of values of k . we write

$$G = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} g_k(x) \sin(k y) dk.$$

The limiting process is introduced because of the k restrictions in the completeness relation in **(b)**. We'll see that the limit is easily executed at the end.

It is sufficient to use $k > 0$ since the antisymmetry of the $\sin(ky)$ in k would just lead to a redefinition of $g_k(x)$ if we were to make the range $-\infty$ to ∞ (leaving out the bit about $k = 0$.) Next, to solve boundary conditions in x and satisfy Laplace's equation where appropriate, we take

$$g_k(x) = A(k) \begin{pmatrix} \sinh(k(d/2 + x)) \sinh(k(d/2 - x')), & \text{for } -d/2 \leq x \leq x' \\ \sinh(k(d/2 + x')) \sinh(k(d/2 - x)), & \text{for } x' \leq x \leq d/2 \end{pmatrix}.$$

As it must, this is constructed from the real exponentials with the separation constant k , it vanishes at $x = \pm d/2$, it is continuous at $x = x'$, and it has a scale factor $A(k)$ to let us normalize the solution to that of a unit point charge. Then we put the G into Poisson's equation to get

$$\nabla^2 G = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} (g_k'' - k^2 g_k) \sin(k y) dk = -\frac{1}{\epsilon_0} \delta(x - x') \delta(y - y') \quad \text{eq. 1}$$

To dissolve the integral into a single term, we use the completeness relation from part **b)**.

Multiply eq. 1 by $\sin(k' y)$, $k' > 0$, to get

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} (g_k'' - k^2 g_k) \sin(k y) \sin(k' y) dk = -\frac{1}{\epsilon_0} \delta(x - x') \delta(y - y') \sin(k' y),$$

then integrate over y to produce

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} (g_k'' - k^2 g_k) \frac{\pi}{2} \delta(k - k') dk = (g_{k'}'' - k'^2 g_{k'}) \frac{\pi}{2} = -\frac{1}{\epsilon_0} \delta(x - x') \sin(k' y').$$

Of course, the k' is just the name of a variable, and so we may call it k instead. Make this substitution and integrate over x covering x' to get

$$\begin{aligned} & \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \frac{\pi}{2} (g_k'(x' + \epsilon) - g_k'(x' - \epsilon)) \\ &= \frac{\pi}{2} k A_k (-\sinh(k(d/2 + x')) \cosh(k(d/2 - x')) - \cosh(k(d/2 + x')) \sinh(k(d/2 - x'))) \\ &= -\frac{1}{\epsilon_0} \sin(k y'). \end{aligned}$$

Simplify to get

$$\frac{\pi}{2} k A_k \sinh(k d) = \frac{1}{\epsilon_0} \sin(k y').$$

So we have an integral representation of the Green function

$$G = \frac{2}{\pi \epsilon_0} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} \frac{1}{k \sinh(k d)} \sinh(k(d/2 + x_{<})) \sinh(k(d/2 - x_{>})) \sin(k y) \sin(k y') dk.$$

Since there is no singularity in the integrand as $k \rightarrow 0$, we can take the limit to get the Green function

$$G = \frac{2}{\pi \epsilon_0} \int_0^{\infty} \frac{1}{k \sinh(k d)} \sinh(k(d/2 + x_{<})) \sinh(k(d/2 - x_{>})) \sin(k y) \sin(k y') dk.$$

Compare these two forms numerically with the given closed expression for some points.

First the sum:

```

Off[NIntegrate::"ploss", NIntegrate::"ncvb"]

pointsnoy = {d → 1, x → .2, xp → -.2, yp → .7};

yvalues = {0, .1, .2, .4, .6, .8, 1., 1.5, 2., 3., 4.};

Gsum =
  With[{y = #}, Sum[
    (
      
$$\frac{2}{\pi} \frac{1}{n} \text{If}[y > y_p, e^{-n\pi y/d} \text{Sinh}[n\pi \frac{y_p}{d}], e^{-n\pi y_p/d} \text{Sinh}[n\pi \frac{y}{d}]$$

      Sin[
        
$$n\pi \left(\frac{x}{d} + \frac{1}{2}\right) \text{Sin}\left[n\pi \left(\frac{x_p}{d} + \frac{1}{2}\right)\right] /. \text{pointsnoy}$$

      ] & /@ yvalues;

Gint = With[{y = #},
  
$$\frac{2}{\pi} \text{NIntegrate}\left[\left(\frac{1}{k \text{Sinh}[k d]} \text{If}[x > x_p, \text{Sinh}[k(d/2 - x)] \text{Sinh}[k(d/2 + x_p)], \right.$$

  
$$\left. \text{Sinh}[k(d/2 - x_p)] \text{Sinh}[k(d/2 + x)]\right) \text{Sin}[k y] \text{Sin}[k y_p]\right] /. \text{pointsnoy}, \{k, 0, 100.\}] \& /@ yvalues;

Gclosed =
  With[{y = #}, 
$$\frac{-1}{4\pi} \text{Log}[\left(\left(\text{Sin}[\pi x/d] \text{Cosh}[\pi y/d] - \text{Sin}[\pi x_p/d] \text{Cosh}[\pi y_p/d]\right)^2 + \right.$$

  
$$\left. \left(\text{Cos}[\pi x/d] \text{Sinh}[\pi y/d] - \text{Cos}[\pi x_p/d] \text{Sinh}[\pi y_p/d]\right)^2\right) / \left(\left(\text{Sin}[\pi x/d] \text{Cosh}[\pi y/d] - \text{Sin}[\pi x_p/d] \text{Cosh}[\pi y_p/d]\right)^2 + \right.$$

  
$$\left. \left(\text{Cos}[\pi x/d] \text{Sinh}[\pi y/d] + \text{Cos}[\pi x_p/d] \text{Sinh}[\pi y_p/d]\right)^2\right)] /. \text{pointsnoy}] \& /@ yvalues;$$$$

```

The following table was computed with $d \rightarrow 1$, $x \rightarrow .2$, $x' \rightarrow -.2$, $y' \rightarrow .7$ and the listed values of y .

```
TableForm[Transpose[{yvalues, Gclosed, Gsum, Gint}],
  TableHeadings -> {None, {"y", "Gclosed", "Gsum", "Gint"}}]

```

y	Gclosed	Gsum	Gint
0	0	0	0.
0.1	0.0124043	0.0124043	0.0124043
0.2	0.0253268	0.0253268	0.0253268
0.4	0.0533554	0.0533554	0.0533554
0.6	0.0775288	0.0775288	0.0775288
0.8	0.0791363	0.0791363	0.0791363
1.	0.0587929	0.0587929	0.0587929
1.5	0.0157282	0.0157282	0.0157282
2.	0.00342422	0.00342422	0.00342422
3.	0.000149659	0.000149659	0.000149659
4.	6.47052×10^{-6}	6.47052×10^{-6}	6.4705×10^{-6}

These restricted numerical calculations give considerable confidence in the equivalence of the three representations.

■ Problem 18

Find the potential inside a long straight segment of a round pipe of radius a . Using two dimensional polar coordinates (ρ, φ) , let the volume be defined by $0 < \rho < a$, and $0 \leq \varphi \leq \varphi_0$ where $\varphi_0 > 0$. The boundary conditions are that $\Phi(0 < \rho \leq a, \varphi = 0) = \Phi(\rho = a, 0 \leq \varphi < \varphi_0) = 0$ and $\Phi(0 < \rho \leq a, \varphi = \varphi_0) = V$. Do not use the Green function constructed with oscillatory functions in the φ variable; I want you to use solutions with oscillatory solutions in the ρ direction.

Solution Problem 18

Since there is no lower limit to ρ , a continuum of k values is possible so make the ansatz

$$\Phi(\rho, \varphi) = \int_{-\infty}^{\infty} A(k) \sinh k\varphi \sin(k \log \frac{\rho}{a}) dk.$$

This satisfies the boundary that Φ vanishes on the boundaries $\varphi = 0$ and $\rho = a$, and of course, it satisfies Laplace's equation for any value of k .

$$\begin{aligned} \text{In}[153]:= & \frac{1}{\rho} \partial_{\rho} \left(\rho \partial_{\rho} \left(\text{Sinh}[k \varphi] \text{Sin}\left[k \text{Log}\left[\frac{\rho}{a}\right]\right] \right) \right) + \\ & \frac{1}{\rho^2} \partial_{\varphi, \varphi} \left(\text{Sinh}[k \varphi] \text{Sin}\left[k \text{Log}\left[\frac{\rho}{a}\right]\right] \right) // \text{FullSimplify} \end{aligned}$$

Out[153]= 0

So what is the function $A(k)$? Choose it so that $V = \int_0^{\infty} A(k) \sinh k\varphi_0 \sin(k \log \frac{\rho}{a}) dk$. Note that we can restrict k to positive values (or set $A(k) = 0$ for $k < 0$) since the function $\sinh k\varphi_0 \sin(k \log \frac{\rho}{a})$ is even in k and so nothing is added by including negative values of k . Clearly we need a completeness relation to dissolve the integral. The

range of the integration should be from $-\infty$ to 0 since that is the range of $\xi = \log \frac{\rho}{a}$ as ρ runs from 0 to a . So consider for, both $k > 0$ and $k' > 0$,

$$\begin{aligned} & \int_{-\infty}^0 \sin k\xi \sin k'\xi \, d\xi \\ &= -\frac{1}{4} \int_{-\infty}^0 (e^{ik\xi} - e^{-ik\xi})(e^{ik'\xi} - e^{-ik'\xi}) \, d\xi \\ &= -\frac{1}{4} \int_{-\infty}^0 (e^{ik\xi} e^{ik'\xi} - e^{-ik\xi} e^{ik'\xi} - e^{ik\xi} e^{-ik'\xi} + e^{-ik\xi} e^{-ik'\xi}) \, d\xi \\ &= -\frac{1}{4} \left(\int_{-\infty}^0 (e^{ik\xi} e^{ik'\xi} - e^{-ik\xi} e^{ik'\xi}) - \int_0^{\infty} (e^{-ik\xi} e^{ik'\xi} - e^{ik\xi} e^{ik'\xi}) \, d\xi \right) \\ &= -\frac{1}{4} \int_{-\infty}^{\infty} (e^{ik\xi} e^{ik'\xi} - e^{-ik\xi} e^{ik'\xi}) \, d\xi \\ &= \frac{\pi}{2} (\delta(k - k') - \delta(k + k')) \end{aligned}$$

Since both both $k > 0$ and $k' > 0$, the second δ function may be dropped since its argument never vanishes. Thus, the completeness relation

$$\int_{-\infty}^0 \sin k\xi \sin k'\xi \, d\xi = \frac{\pi}{2} \delta(k - k') \text{ when } k > 0 \text{ and } k' > 0.$$

So multiply both sides of

$$V = \int_0^{\infty} A(k) \sinh k\varphi_0 \sin(k \log \frac{\rho}{a}) \, dk$$

by $\sin(k' \log \frac{\rho}{a}) \, d(\log \frac{\rho}{a}) = \sin k' \xi \, d\xi$ and integrate from $\rho = 0$ to a , or ξ from $-\infty$ to 0, to get

$$\begin{aligned} V \int_{\xi=-\infty}^0 \sin k' \xi \, d\xi &= \int_{\xi=-\infty}^0 \int_{k=0}^{\infty} A(k) \sinh k\varphi_0 \sin(k \xi) \sin k' \xi \, d\xi \, dk \\ &= \frac{\pi}{2} \int_{k=0}^{\infty} A(k) \sinh k\varphi_0 \delta(k - k') \, dk \\ &= \frac{\pi}{2} A(k') \sinh k' \varphi_0. \end{aligned}$$

Lastly, we need $\int_{\xi=-\infty}^0 \sin k' \xi \, d\xi = \text{Im} \int_{\xi=-\infty}^0 e^{ik' \xi} \, d\xi$. As a variation, do this integral with an "integrating factor" by defining it as

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \text{Im} \int_{\xi=-\infty}^0 e^{(ik'+\epsilon)\xi} \, d\xi = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \text{Im} \left(\frac{e^{(ik'+\epsilon)\xi}}{ik'+\epsilon} \Big|_{\xi=-\infty}^0 \right) = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \text{Im} \left(\frac{1}{ik'+\epsilon} \right) = -\frac{1}{k'}.$$

So we get

$$-\frac{V}{k'} = \frac{\pi}{2} A(k') \sinh k' \varphi_0 \quad \Rightarrow \quad A(k) = -\frac{2}{\pi} \frac{V}{k \sinh k \varphi_0}$$

and the representation of the solution as an integral,

$$\Phi(\rho, \varphi) = -\frac{2V}{\pi} \int_0^{\infty} \frac{\sinh k\varphi}{k \sinh k\varphi_0} \sin(k \log \frac{\rho}{a}) \, dk.$$

Because of the treatment of a couple of irregular integrals in the solution, you may need some reassurance that this function really works! Notice that the integrand is well behaved at $k = 0$, and as $k \rightarrow \infty$, the integrand becomes $\frac{1}{k} e^{k(\varphi-\varphi_0)} \sin(k \log \frac{\rho}{a})$ which is zero *on the average*, i.e., over any small range of k for large k , it oscillates so often that its integral can be taken as zero. This behavior, however, makes numerical evaluation problematic and *Mathematica* gives lots of warnings in an evaluation! However, it is not at all hard to get a representation of the solution with a Green function and compare its evaluations with the hard-to-do integral. Just ignore the warnings and look:

```
In[207]:= Plot[Evaluate[
  
$$\frac{-2}{\pi} \text{NIntegrate}\left[\frac{\text{Sinh}[k \#]}{k \text{Sinh}[k \varphi 0]} \text{Sin}\left[k \text{Log}\left[\frac{\rho}{a}\right]\right] /. \{\varphi 0 \rightarrow 30.5 \text{ Degree}, a \rightarrow 1\},\right.$$

  {k, .01, 1000}, MaxRecursion -> 20] & /@ Table[\varphi Degree, {\varphi, 0, 30, 5}],
  {\rho, .05, 1.}, PlotRange -> {{0, 1.1}, {0, 1.1}}, Frame -> True,
  FrameLabel ->
  {"\rho", "\varphi", "Potential from integral for \nseveral fixed angles \varphi", ""},
  ImageSize -> 724]
```

NIntegrate::ploss :

Numerical integration stopping due to loss of precision. Achieved neither the requested PrecisionGoal nor AccuracyGoal; suspect one of the following: highly oscillatory integrand or the true value of the integral is 0. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::inum :

Integrand $\frac{\text{Sinh}[k (5^\circ)] \text{Sin}[\llbracket 1 \rrbracket]}{k \text{Sinh}[k \varphi 0]}$ /. {\varphi 0 -> 30.5°, a -> 1} is not numerical at {k} = {500.005}. More...

NIntegrate::inum :

Integrand $\frac{\text{Sinh}[k (10^\circ)] \text{Sin}[\llbracket 1 \rrbracket]}{k \text{Sinh}[k \varphi 0]}$ /. {\varphi 0 -> 30.5°, a -> 1} is not numerical at {k} = {500.005}. More...

NIntegrate::inum :

Integrand $\frac{\text{Sinh}[k (15^\circ)] \text{Sin}[\llbracket 1 \rrbracket]}{k \text{Sinh}[k \varphi 0]}$ /. {\varphi 0 -> 30.5°, a -> 1} is not numerical at {k} = {500.005}. More...

General::stop : Further output of

NIntegrate::inum will be suppressed during this calculation. More...

NIntegrate::slwcon :

Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration being 0, oscillatory integrand, or insufficient WorkingPrecision. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::slwcon :

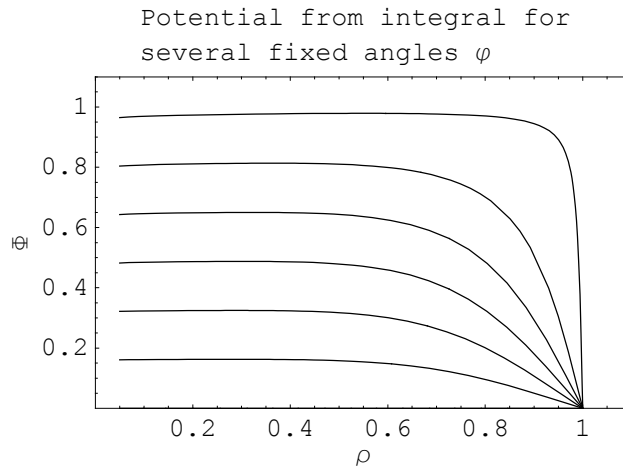
Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration being 0, oscillatory integrand, or insufficient WorkingPrecision. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::slwcon :

Numerical integration converging too slowly; suspect one of the following:
singularity, value of the integration being 0, oscillatory integrand,
or insufficient WorkingPrecision. If your integrand is oscillatory
try using the option Method->Oscillatory in NIntegrate. More...

General::stop : Further output of

NIntegrate::slwcon will be suppressed during this calculation. More...



Out[207]= - Graphics -

As this plot shows, the result of the numerical integration, in spite of various warnings, is perfectly reasonable. Note that the potential at the origin is not defined, as you would expect since it is a singular point of the coordinate system, i.e., covered by every value of φ ; which value of the potential you get there depends upon from which direction you approach the origin.

Now represent the potential in the volume by using the Green function method. This will in fact give us a closed form for the solution with which to compare the integral representation.

■ Green function Approach

As usual, assume

$$G = \sum_{n=1}^{\infty} A_n g_n(\rho, \rho') \sin(k \varphi)$$

where $g_n(z, z') = \begin{cases} \left(\frac{\rho}{a}\right)^k \left(\left(\frac{\rho'}{a}\right)^k - \left(\frac{a}{\rho'}\right)^k\right) & 0 < \rho < \rho' \\ \left(\frac{\rho'}{a}\right)^k \left(\left(\frac{\rho}{a}\right)^k - \left(\frac{a}{\rho}\right)^k\right) & \rho' < \rho < a \end{cases}$ with $k = \frac{n\pi}{\varphi_0}$, is constructed from the two radial functions

ρ^k and ρ^{-k} so that it is zero at the origin and at $\rho = a$, and continuous at $\rho = \rho'$. To be explicit, the separation parameter k should have an index, k_n , but to avoid writings complications I will use the index only when needed for clarity.

Then use $\nabla^2 G = -\frac{1}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi')$ to get

$$\nabla^2 G = \sum_{n=1}^{\infty} A_n \left(\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_n}{d\rho} \right) - \frac{1}{\rho^2} k^2 g_n \right) \sin(k\varphi) = -\frac{1}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi').$$

Multiply both sides by $\sin(k_m \varphi)$ and integrate over φ from 0 to φ_0 to dissolve the sum into a single term;

$$A_m \frac{\varphi_0}{2} \left(\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_m}{d\rho} \right) - \frac{1}{\rho^2} (k_m)^2 g_m \right) = -\frac{1}{\varepsilon_0} \frac{1}{\rho} \delta(\rho - \rho') \sin(k_m \varphi')$$

Let $m \rightarrow n$, multiply by ρ , integrate just over the delta function, and use the continuity of g_n to get

$$A_n \lim_{\epsilon \rightarrow 0} \left(\left(\rho \frac{dg_n}{d\rho} \right)_{\rho=\rho'+\epsilon} - \left(\rho \frac{dg_n}{d\rho} \right)_{\rho=\rho'-\epsilon} \right) = -\frac{1}{\varepsilon_0} \frac{2}{\varphi_0} \sin(k \varphi'),$$

or

$$A_n \rho' k \left(\left(\frac{\rho'}{a} \right)^k \left(\frac{\rho'^{k-1}}{a^k} + \frac{a^k}{\rho'^{k+1}} \right) - \frac{\rho'^{k-1}}{a^k} \left(\left(\frac{\rho'}{a} \right)^k - \left(\frac{a}{\rho'} \right)^k \right) \right) = -\frac{1}{\varepsilon_0} \frac{2}{\varphi_0} \sin(k \varphi').$$

This gives

$$A_n = -\frac{1}{\varepsilon_0} \frac{1}{n\pi} \sin\left(n\pi \frac{\varphi'}{\varphi_0}\right)$$

so the Green function is

$$G(\rho, \varphi; \rho', \varphi') = -\frac{1}{\varepsilon_0} \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\frac{\rho_{<}}{a} \right)^{\frac{n\pi}{\varphi_0}} \left(\left(\frac{\rho_{>}}{a} \right)^{\frac{n\pi}{\varphi_0}} - \left(\frac{a}{\rho_{>}} \right)^{\frac{n\pi}{\varphi_0}} \right) \sin\left(n\pi \frac{\varphi}{\varphi_0}\right) \sin\left(n\pi \frac{\varphi'}{\varphi_0}\right), \quad \left(\rho_{>} = \text{Max}[\rho, \rho'] \right), \\ \left(\rho_{<} = \text{Min}[\rho, \rho'] \right).$$

Since there is no charge density inside the volume, we have from Green's theorem that for the problem at hand (the surface at φ_0 is at potential V and the others are grounded):

$$\begin{aligned} \Phi(\rho, \varphi) &= -\varepsilon_0 \int_{\partial V} \Phi(\rho', \varphi') \nabla' G(\rho, \varphi; \rho', \varphi') \cdot d\vec{A}' \\ &= -\varepsilon_0 V \int_{\rho'=0}^a \frac{1}{\rho'} \frac{\partial}{\partial \varphi'} G(\rho, \varphi; \rho', \varphi') \Big|_{\varphi'=\varphi_0} d\rho' \\ &= \frac{V}{\varphi_0} \sum_{n=1}^{\infty} \sin\left(n\pi \frac{\varphi}{\varphi_0}\right) \cos\left(n\pi \frac{\varphi_0}{\varphi_0}\right) \left(\left(\frac{\rho}{a} \right)^{\frac{n\pi}{\varphi_0}} - \left(\frac{a}{\rho} \right)^{\frac{n\pi}{\varphi_0}} \right) \int_{\rho'=0}^{\rho} \frac{\rho'^{\frac{n\pi}{\varphi_0}-1}}{a^{\frac{n\pi}{\varphi_0}}} d\rho' + \left(\frac{\rho}{a} \right)^{\frac{n\pi}{\varphi_0}} \int_{\rho'=\rho}^a \left(\frac{\rho'^{\frac{n\pi}{\varphi_0}-1}}{a^{\frac{n\pi}{\varphi_0}}} - \frac{a^{\frac{n\pi}{\varphi_0}}}{\rho'^{\frac{n\pi}{\varphi_0}+1}} \right) d\rho' \end{aligned}$$

Do the trivial but tedious integrals with *Mathematica*

In[201]:= **Assuming** [{ $\rho \in \text{Reals}$, $a \in \text{Reals}$, $a > \rho$, $k > 0$, $\rho > 0$ } ,

$$\left(\left(\left(\frac{\rho}{a} \right)^k - \left(\frac{a}{\rho} \right)^k \right) \int_0^{\rho} \frac{\mathbf{x}^{k-1}}{\mathbf{a}^k} d\mathbf{x} + \left(\frac{\rho}{a} \right)^k \int_{\rho}^a \left(\frac{\mathbf{x}^{k-1}}{\mathbf{a}^k} - \frac{\mathbf{a}^k}{\mathbf{x}^{k+1}} \right) d\mathbf{x} \right) // \text{FullSimplify}$$

$$\text{Out[201]} = \frac{2 \left(-1 + \left(\frac{\rho}{a} \right)^k \right)}{k}$$

So you get

$$\Phi(\rho, \varphi) = V \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(n\pi \frac{\varphi}{\varphi_0}\right) (-1)^{n+1} \left(1 - \left(\frac{\rho}{a} \right)^{\frac{n\pi}{\varphi_0}} \right)$$

At first sight it seems manifest that this function cannot be the solution of the given problem since clearly $\Phi(\rho, \varphi_0) = 0$ whereas the boundary condition is that the potential is to be V on this surface. The resolution of this paradox is the fact that Fourier's infinite series, $\sum_{n=-\infty}^{\infty} F_n e^{i2\pi n \frac{x}{\tau}}$ where $F_n = \frac{1}{\tau} \int_0^{\tau} f(x) e^{-i2\pi n \frac{x}{\tau}} dx$, does

NOT sum to $f(x)$ but rather to $\lim_{\epsilon \rightarrow 0} (f(x - \epsilon) + f(x + 1))$. Thus if $f(x)$ is discontinuous at a point x , then the value of the series is the average value of the two values at x . For the case at hand, it is clear that the series is an odd function in φ about $\varphi = \varphi_0$ and so the series will sum to zero exactly at $\varphi = \varphi_0$ which is what we see.

Clearly, the quantity of interest for the potential is $\lim_{\epsilon \rightarrow 0} V \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi \frac{\varphi_0 - \epsilon}{\varphi_0}) (-1)^{n+1} (1 - (\frac{\rho}{a})^{\frac{n\pi}{\varphi_0}})$ which should be V .

Using

```
In[235]:= Series[-Log[1 + x], {x, 0, 5}]
```

```
Out[235]= -x + x^2/2 - x^3/3 + x^4/4 - x^5/5 + O[x]^6
```

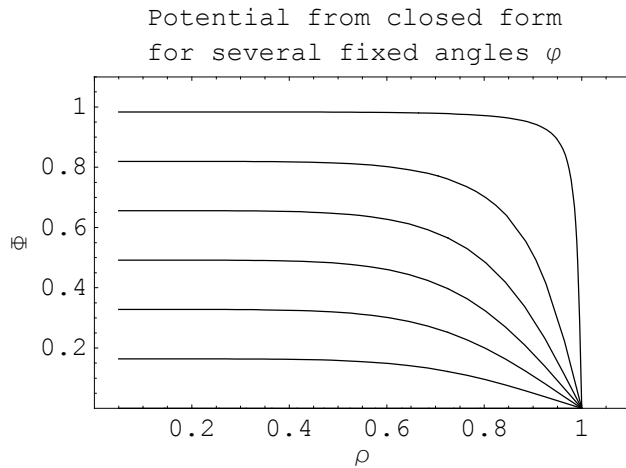
we can sum the series by writing

$$\begin{aligned} \Phi(\rho, \varphi) &= -\frac{2V}{\pi} \operatorname{Im} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \left(\left(e^{i\pi \frac{\varphi}{\varphi_0}} \right)^n - \left(e^{i\pi \frac{\varphi}{\varphi_0}} \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right)^n \right) \\ &= \frac{2V}{\pi} \operatorname{Im} \log \frac{1 + e^{i\pi \frac{\varphi}{\varphi_0}}}{1 + e^{i\pi \frac{\varphi}{\varphi_0}} \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}}} \\ &= \frac{2V}{\pi} \operatorname{Im} \log \left(1 + e^{i\pi \frac{\varphi}{\varphi_0}} \right) \frac{\left(1 + e^{-i\pi \frac{\varphi}{\varphi_0}} \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right)}{\left(\left| 1 + e^{i\pi \frac{\varphi}{\varphi_0}} \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right| \right)^2} \\ &= \frac{2V}{\pi} \operatorname{Im} \log \frac{1 + \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} + \cos\left(\pi \frac{\varphi}{\varphi_0}\right) \left(1 + \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right) + i \sin\left(\pi \frac{\varphi}{\varphi_0}\right) \left(1 - \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right)}{\left(\left| 1 + e^{i\pi \frac{\varphi}{\varphi_0}} \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right| \right)^2} \\ &= \frac{2V}{\pi} \arctan \left(\frac{\sin\left(\pi \frac{\varphi}{\varphi_0}\right) \left(1 - \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right)}{\left(1 + \cos\left(\pi \frac{\varphi}{\varphi_0}\right) \right) \left(1 + \left(\frac{\rho}{a} \right)^{\frac{\pi}{\varphi_0}} \right)} \right). \end{aligned}$$

This clearly satisfies the boundary condition at $\varphi = 0$ and $\rho = a$. At $\varphi = \varphi_0$ we have a 0/0 problem so use L'Hospital's rule to get $\frac{2V}{\pi} \frac{\pi}{2} = V$ and we see that the "problem" at $\varphi = \varphi_0$ has vanished. Since the function is the imaginary part of an analytic complex function, it satisfies the Cauchy-Riemann equations, which is just the Laplace equation. From uniqueness, this is a representation of THE solution.

So take a look at plots of it.

```
In[237]:= Plot[Evaluate[With[{φ = #}, V  $\frac{2}{\pi}$  ArcTan[ $\frac{\text{Sin}[\pi \frac{\varphi}{\varphi_0}] (1 - (\frac{\rho}{a})^{\frac{\pi}{\varphi_0}})}{(1 + \text{Cos}[\pi \frac{\varphi}{\varphi_0}]) (1 + (\frac{\rho}{a})^{\frac{\pi}{\varphi_0}})}$ ]] /.
    {φ0 → 30.5 Degree, a → 1, V → 1}] & /@ Table[φ Degree, {φ, 0, 30, 5}],
    {ρ, .05, 1.}, PlotRange → {{0, 1.1}, {0, 1.1}}, Frame → True,
    FrameLabel →
    {"ρ", "Φ", "Potential from closed form \nfor several fixed angles φ", ""}]
```



Out[237]= - Graphics -

Clearly this agrees with the plot we got from the integral representation of the solution. So we get the rather amusing mathematical equality:

$$\int_0^\infty \frac{\sinh k\varphi}{k \sinh k\varphi_0} \sin\left(k \log \frac{\rho}{a}\right) dk = -\arctan\left(\frac{\sin\left(\pi \frac{\varphi}{\varphi_0}\right) \left(1 - \left(\frac{\rho}{a}\right)^{\frac{\pi}{\varphi_0}}\right)}{\left(1 + \cos\left(\pi \frac{\varphi}{\varphi_0}\right)\right) \left(1 + \left(\frac{\rho}{a}\right)^{\frac{\pi}{\varphi_0}}\right)}\right)$$

■ Problem 19

Find the Green function for the inside of a rectangular box with dimensions a in the x -direction, b in the y -direction, and c in the z -direction. For convenience, let one of the corners of the box be at the origin - use figure 2.9 of Jackson to define the coordinate system. Further, use real exponential functions in the z coordinate in your answer. Of course, it is easy then to write other representations of the solution with the real exponentials in the x or y direction. **Do not use the method of images for this problem.**

Solution Problem 19

In three dimensional cartesian coordinates, choose as solutions sums over the separation constants k_x and k_y , products of sinusoids in $k_x x$, sinusoids in $k_y y$, and sums of exponentials in $k z$, where $k = \sqrt{k_x^2 + k_y^2}$. To satisfy the Green function boundary conditions we need $k_x = n\pi \frac{1}{a}$, $k_y = m\pi \frac{1}{b}$ where n and m are positive integers and the functional form is

$$G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g_{nm}(z) \sin(n \pi \frac{x}{a}) \sin(m \pi \frac{y}{b}).$$

$$\text{with } g_{nm}(z) = A_{nm} \begin{pmatrix} \sinh(k_{nm} z) \sinh(k_{nm}(c - z')) & , \text{ for } 0 \leq z \leq z' \\ \sinh(k_{nm} z') \sinh(k_{nm}(c - z)) & , \text{ for } z' \leq z \leq c \end{pmatrix}, \quad \text{with } k_{nm} = \pi \sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}.$$

This function is arranged so that G satisfies the Laplace equation where appropriate, satisfies the boundary conditions at $z = 0$ and c , and is continuous at $z = z'$. We now adjust the constant A_{nm} so that Poisson's equation is satisfied for a point charge at (x', y', z') .

$$\begin{aligned} \nabla^2 G &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (g''_{nm}(z) - (\frac{n\pi}{a})^2 g_{nm}(z) - (\frac{m\pi}{b})^2 g_{nm}(z)) \sin(n \pi \frac{x}{a}) \sin(m \pi \frac{y}{b}) \\ &= -\frac{1}{\epsilon_0} \delta(x - x') \delta(y - y') \delta(z - z'). \end{aligned}$$

Now multiply both sides by $\sin(n' \pi \frac{x}{a}) \sin(m' \pi \frac{y}{b})$ and integrate on x from 0 to a and on y from 0 to b . The orthogonality of the sinusoids dissolves the sum into a single term in which I replace (n', m') , which are just variable names, by (n, m) .

$$\begin{aligned} \frac{ab}{4} (g''_{nm}(z) - (\frac{n\pi}{a})^2 g_{nm}(z) - (\frac{m\pi}{b})^2 g_{nm}(z)) \\ = -\frac{1}{\epsilon_0} \sin(n \pi \frac{x'}{a}) \sin(m \pi \frac{y'}{b}) \delta(z - z'). \end{aligned}$$

Lastly, integrate over z from $z' - \epsilon$ to $z' + \epsilon$, $\epsilon > 0$, and take the limit as $\epsilon \rightarrow 0$. This gives

$$\begin{aligned} \frac{ab}{4} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} (g'_{nm}(z' + \epsilon) - g'_{nm}(z' - \epsilon)) \\ = \frac{ab}{4} A_{nm} k_{nm} (-\sinh(k_{nm} z') \cosh(k_{nm}(c - z')) - \cosh(k_{nm} z') \sinh(k_{nm}(c - z'))) \\ = -\frac{ab}{4} A_{nm} k_{nm} \sinh(k_{nm} c) \\ = -\frac{1}{\epsilon_0} \sin(n \pi \frac{x'}{a}) \sin(m \pi \frac{y'}{b}), \end{aligned}$$

$$\text{or, } A_{nm} = \frac{4}{ab \epsilon_0} \frac{1}{k_{nm} \sinh(k_{nm} c)} \sin(n \pi \frac{x'}{a}) \sin(m \pi \frac{y'}{b}).$$

Letting $z_>$ be the greater of z and z' and $z_<$ the smaller of the two, we get for the Green function

$$G = \frac{4}{ab \epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh(k_{nm} z_<) \sinh(k_{nm}(c - z_>))}{k_{nm} \sinh(k_{nm} c)} \sin(n \pi \frac{x}{a}) \sin(n \pi \frac{x'}{a}) \sin(m \pi \frac{y}{b}) \sin(m \pi \frac{y'}{b}).$$

■ Problem 20

Find the force acting in the z direction on a point charge of magnitude q located at point (x', y', z') inside the box given in problem 19 if all of its walls are at zero potential.

Hint: Be sure to realize that the Green function found in problem 19 is singular at the point

$\vec{r} = (x, y, z) = (x', y', z') = \vec{r}'$ and so the z force on q is *not* $-q^2 \frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$ because G contains contributions from both the direct field of the point charge and that due to the induced charges in the walls of the box.

Either

a) don't forget that Newton's third law applies in electrostatics,

or

b) notice that the singularity in $\frac{\partial G(\vec{r}; \vec{r}')}{\partial z} \Big|_{\vec{r}=\vec{r}'}$ can be removed by considering

$$\frac{1}{2} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \left(-\frac{\partial G(x', y', z' - \epsilon; x', y', z')}{\partial z} - \frac{\partial G(x', y', z' + \epsilon; x', y', z')}{\partial z} \right).$$

Solution Problem 20

Find the z component of the force on a charge q in the box of the last problem if its walls are all grounded. The potential in this case is just qG where G is the Green function from the last problem. This force comes from the induced charges on the inside walls of the box, of course.

a) First use Newton's third law which says that the force on the charge is equal and opposite to the force on the box. This latter is just the integral over the inside area of the force on the induced charge density and this is $\int_{\text{inside surface}} \frac{\epsilon_0}{2} |\vec{\nabla}(qG) \cdot \vec{n}|^2 \vec{n} dA$ where \vec{n} is a vector on the inside wall of the box pointing into the box. For the z component of force on the charge, F_z , we only need the surfaces at $z = 0$ and $z = c$ since the forces on the other faces do not have a component in the z direction. So we have, using

$$(\sum_n a_n)^2 = \sum_n \sum_{n'} a_n a_{n'},$$

$$\begin{aligned} F_z &= q^2 \int_0^a dx \int_0^b dy \frac{\epsilon_0}{2} \left(\left(\frac{\partial G}{\partial z} \Big|_{z=c} \right)^2 - \left(\frac{\partial G}{\partial z} \Big|_{z=0} \right)^2 \right) \\ &= q^2 \frac{\epsilon_0}{2} \left(\frac{4}{ab\epsilon_0} \right)^2 \int_0^a dx \int_0^b dy \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=1}^{\infty} \left(\frac{\sinh(k_{nm} z')}{\sinh(k_{nm} c)} \frac{\sinh(k_{n'm'} z')}{\sinh(k_{n'm'} c)} \right. \\ &\quad \left. - \frac{\sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \frac{\sinh(k_{n'm'}(c-z'))}{\sinh(k_{n'm'} c)} \right) \sin(n\pi \frac{x}{a}) \sin(n\pi \frac{x'}{a}) \\ &\quad \sin(m\pi \frac{y}{b}) \sin(m\pi \frac{y'}{b}) \sin(n'\pi \frac{x}{a}) \sin(n'\pi \frac{x'}{a}) \sin(m'\pi \frac{y}{b}) \sin(m'\pi \frac{y'}{b}). \end{aligned}$$

Now integrate and use orthogonality to dissolve the sums over n' and m' .

$$\begin{aligned}
F_z &= q^2 \frac{\epsilon_0}{2} \left(\frac{4}{ab\epsilon_0} \right)^2 \frac{ab}{4} \\
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\sinh(k_{nm} z')}{\sinh(k_{nm} c)} \frac{\sinh(k_{nm} z')}{\sinh(k_{nm} c)} - \frac{\sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \frac{\sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \right) \sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \\
&= q^2 \frac{2}{ab\epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\left(\frac{\sinh(k_{nm} z')}{\sinh(k_{nm} c)} \right)^2 - \left(\frac{\sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \right)^2 \right) \left(\sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \right)^2. \\
&\quad \mathbf{Sinh[k z]^2 - Sinh[k (c - z)]^2 // TrigExpand // Simplify} \\
&\quad -Sinh[c k] Sinh[k (c - 2 z)]
\end{aligned}$$

With this simplification,

$$F_z = q^2 \frac{2}{ab\epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh(2k_{nm}(z' - \frac{c}{2}))}{\sinh(k_{nm} c)} \left(\sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \right)^2.$$

b) Now let's get the force by using the idea of writing that it is just $F_z = q E_z^{\text{due to induced charge only}}$. The difficulty is that the electric field obtained from $-q(\nabla G)_z$ yields $E_z^{\text{total}} = E_z^{\text{due to } q \text{ itself}} + E_z^{\text{due to induced charge only}}$. However, it is easy to get rid of the field due to q directly by noting that it is equal in magnitude but oppositely directed on symmetrically placed points on each side of q , by Coulomb's law directly, whereas the contribution due to the induced charge does not act like this. In fact if the two points are very near the position of the charge, the contribution of the induced charge is essentially the same at the two points. Thus we can get what we want from

$$F_z = \frac{1}{2} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \left(-q^2 \frac{\partial G}{\partial z} (x', y', z' + \epsilon; x', y', z') - q^2 \frac{\partial G}{\partial z} (x', y', z' - \epsilon; x', y', z') \right),$$

or

$$F_z = -q^2 \frac{2}{ab\epsilon_0} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-\frac{\sinh(k_{nm} z') \cosh(k_{nm}(c-(z'+\epsilon)))}{\sinh(k_{nm} c)} + \frac{\cosh(k_{nm}(z'-\epsilon)) \sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \right) \left(\sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \right)^2.$$

$$\begin{aligned}
\text{But } &-\frac{\sinh(k_{nm} z') \cosh(k_{nm}(c-(z'+\epsilon)))}{\sinh(k_{nm} c)} + \frac{\cosh(k_{nm}(z'-\epsilon)) \sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} \rightarrow \\
&-\frac{\sinh(k_{nm} z') \cosh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} + \frac{\cosh(k_{nm} z') \sinh(k_{nm}(c-z'))}{\sinh(k_{nm} c)} = \frac{\sinh(k_{nm}(c-2z'))}{\sinh(k_{nm} c)}
\end{aligned}$$

as $\epsilon \rightarrow 0$ so that

$$\begin{aligned}
F_z &= -q^2 \frac{2}{ab\epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh(k_{nm}(c-2z'))}{\sinh(k_{nm} c)} \left(\sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \right)^2 \\
&= q^2 \frac{2}{ab\epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh(2k_{nm}(z' - \frac{c}{2}))}{\sinh(k_{nm} c)} \left(\sin(n\pi \frac{x'}{a}) \sin(m\pi \frac{y'}{b}) \right)^2
\end{aligned}$$

as above.

It is interesting to see how this rather complicated formula contains within itself the result that for small z' and much smaller than either x' or y' , we can get an approximation to the force from just one image charge, the one at $(x', y', -z')$ which is $F_z \approx -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2z')^2}$. It is easy to see that in this case the series converges very slowly since each of its terms is positive, but they do decrease slowly as n and m increase. For large k_{nm} , the summands approach (remember that $z' \ll c$) $-e^{k_{nm}(c-2z')}/e^{k_{nm}c} = -e^{-2k_{nm}z'}$. So the series converges, but the smaller z' , the more slowly. So most of the contribution of the sum comes from large values of n and m where $k_x \equiv \frac{n\pi}{a}$ is much larger than $\Delta k_x = \frac{\pi}{a}$. Similarly for k_y and Δk_y . So we can write

$$F_z = q^2 \frac{2}{\pi^2 \epsilon_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh(2k(z' - \frac{c}{2}))}{\sinh(kc)} (\sin(k_x x') \sin(k_y y'))^2 \Delta k_x \Delta k_y$$

where $k = \sqrt{k_x^2 + k_y^2}$. Approximating the sum as an integral gives

$$F_z \approx q^2 \frac{2}{\pi^2 \epsilon_0} \int_0^{\infty} \int_0^{\infty} e^{-2kz'} (\sin(k_x x') \sin(k_y y'))^2 dk_x dk_y.$$

Next, notice that you won't make much of an error by replacing the squares of the sinusoids by their average values of $\frac{1}{2}$ so the next outrage is to write

$$F_z \approx q^2 \frac{1}{2\pi^2 \epsilon_0} \int_0^{\infty} \int_0^{\infty} e^{-2kz'} dk_x dk_y = q^2 \frac{1}{2\pi^2 \epsilon_0} \int_0^{\infty} \int_0^{\pi/2} e^{-2kz'} k dk d\varphi_k = \frac{q^2}{4\pi \epsilon_0} \int_0^{\infty} e^{-2kz'} k dk.$$

$$\text{Simplify} \left[\int_0^{\infty} e^{-2kz'} k dk, z' > 0 \right]$$

$$\frac{1}{4(z')^2}$$

And so you get the approximation when $z' \ll c$, $z' \ll x'$, $z' \ll y'$ that

$$F_z \approx \frac{1}{4\pi \epsilon_0} \frac{q^2}{(2z')^2}$$

as expected.

■ Problem 21 Jackson, problem 2.13

(a) Two halves of a long hollow conducting cylinder of radius b are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2 . Show that the potential inside is given by

$$\Phi(\rho, \varphi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1} \left(\frac{2b\rho}{b^2 - \rho^2} \cos \varphi \right)$$

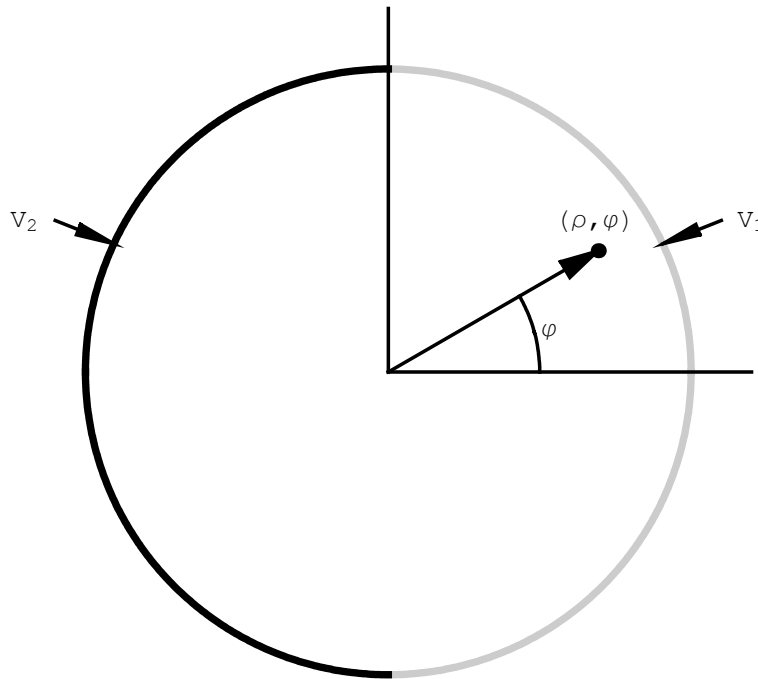
where φ is measured from a plane perpendicular to the plane through the gaps.

(b) Calculate the surface charge density on each half of the cylinder.

Begin graphics

End graphics

Show [fig213]



- Graphics -

Solution Problem 21

(a) Using the sinusoidal solutions in φ , we can immediately write

$$\Phi = \sum_{n=0}^{\infty} A_n \left(\frac{\rho}{b}\right)^n \cos n\varphi.$$

This form is chosen so that the potential is regular at the origin, is continuous at $\varphi = 0$, and it builds in the symmetry of the boundary condition, i.e., that it is symmetric under $\varphi \rightarrow -\varphi$. Also since the only scale in the problem is the radius b we may as well express the radial coordinate in it as the unit. Finally, we need to choose the A_n so that

$$\sum_{n=0}^{\infty} A_n \cos n\varphi = f(\varphi) \quad \text{where} \quad f(\varphi) = \begin{cases} V_1 & -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \\ V_2 & \frac{\pi}{2} < \varphi < \frac{3\pi}{2} \end{cases}.$$

The orthogonality of the sinusoids give $\int_{-\pi/2}^{3\pi/2} \cos n\varphi \cos m\varphi d\varphi = \pi \delta_{nm}$. So we get for $m \neq 0$

$$\pi A_m = \int_{-\pi/2}^{3\pi/2} f(\varphi) \cos m\varphi d\varphi = V_1 \int_{-\pi/2}^{\pi/2} \cos m\varphi d\varphi + V_2 \int_{\pi/2}^{3\pi/2} \cos m\varphi d\varphi = \frac{2(V_1 - V_2)}{m} \begin{pmatrix} 0 & m \text{ even} \\ (-1)^{\frac{m-1}{2}} & m \text{ odd} \end{pmatrix}.$$

A simple integral gives $A_0 = \frac{1}{2\pi} (\pi V_1 + \pi V_2)$.

Finally then, the solution can be expressed in a series as

$$\begin{aligned}\Phi &= \frac{V_1+V_2}{2} + 2 \frac{V_1-V_2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\rho}{b}\right)^{2n+1} \cos((2n+1)\varphi) \\ &= \frac{V_1+V_2}{2} + 2 \frac{V_1-V_2}{\pi} \operatorname{Re}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\rho}{b}\right)^{2n+1} e^{i(2n+1)\varphi}\right) \\ &= \frac{V_1+V_2}{2} + 2 \frac{V_1-V_2}{\pi} \operatorname{Re}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\rho}{b} e^{i\varphi}\right)^{2n+1}\right).\end{aligned}$$

The power series is summable:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

ArcTan[x]

$$\text{So } \Phi = \frac{V_1+V_2}{2} + 2 \frac{V_1-V_2}{\pi} \operatorname{Re}(\tan^{-1}(\frac{\rho}{b} e^{i\varphi}))$$

$$\text{Now use } y = \tan^{-1} x = \frac{1}{2i} \log\left(\frac{1+ix}{1-ix}\right)$$

to get

$$\operatorname{Re}(\tan^{-1} x) = \frac{1}{2} \arg\left(\frac{1+ix}{1-ix}\right) = \frac{1}{2} \tan^{-1}\left(\frac{\operatorname{Im}\left(\frac{1+ix}{1-ix}\right)}{\operatorname{Re}\left(\frac{1+ix}{1-ix}\right)}\right) = \frac{1}{2} \tan^{-1}\left(\frac{\operatorname{Im}((1+ix)(1+ix^*))}{\operatorname{Re}((1+ix)(1+ix^*))}\right).$$

So

$$\begin{aligned}\operatorname{Re}(\tan^{-1}(\frac{\rho}{b} e^{i\varphi})) &= \operatorname{Re}\left(\frac{1}{2i} \log\left(\frac{1+i\frac{\rho}{b} e^{i\varphi}}{1-i\frac{\rho}{b} e^{i\varphi}}\right)\right) = \left(\frac{1}{2} \tan^{-1}\left(\frac{\operatorname{Im}((1+i\frac{\rho}{b} e^{i\varphi})(1+i\frac{\rho}{b} e^{-i\varphi}))}{\operatorname{Re}((1+i\frac{\rho}{b} e^{i\varphi})(1+i\frac{\rho}{b} e^{-i\varphi}))}\right)\right) \\ &= \frac{1}{2} \tan^{-1} \frac{2\frac{\rho}{b} \cos\varphi}{1-(\frac{\rho}{b})^2} = \frac{1}{2} \tan^{-1}\left(\frac{2\rho b \cos\varphi}{b^2-\rho^2}\right).\end{aligned}$$

This then gives the result in a nice form:

$$\Phi = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi} \tan^{-1}\left(\frac{2\rho b \cos\varphi}{b^2-\rho^2}\right).$$

Do a numerical check to see that no factors of 2 or π have gone amissing.

$$\text{With}\left[\left\{\mathbf{r} = \#, \varphi = 27 \text{ Degree}\right\}, \left\{\operatorname{ArcTan}\left[\frac{2 \mathbf{r} \operatorname{Cos}[\varphi]}{1 - \mathbf{r}^2}\right]\right\},\right.$$

$$\left.2 \sum_{n=0}^{100} \frac{(-1)^n}{2n+1} \mathbf{r}^{2n+1} \operatorname{Cos}[(2n+1)\varphi]\right] \& /@ \text{Table}[\mathbf{r}, \{\mathbf{r}, .1, .9, .1\}]$$

$$\{\{0.178094, 0.178094\}, \{0.355481, 0.355481\}, \{0.531116, 0.531116\},$$

$$\{0.703668, 0.703668\}, \{0.871114, 0.871114\}, \{1.03143, 1.03143\},$$

$$\{1.18269, 1.18269\}, \{1.32344, 1.32344\}, \{1.45288, 1.45288\}\}$$

Looks reasonable.

(b) To calculate the surface charge density we use $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{charge}}}{\epsilon_0}$ at the inside surface of the cylinder to get

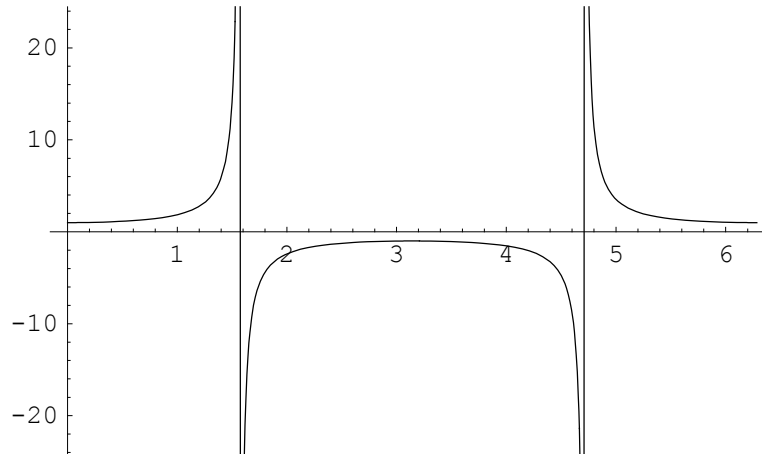
$$\sigma = -\epsilon_0 E_\rho \Big|_{\rho=b} = \epsilon_0 \frac{\partial \Phi}{\partial \rho} \Big|_{\rho=b} = \epsilon_0 \frac{V_1-V_2}{\pi} \frac{\partial}{\partial \rho} \tan^{-1}\left(\frac{2\rho b \cos\varphi}{b^2-\rho^2}\right) \Big|_{\rho=b}$$

$$\text{Simplify@}\left(\partial_{\rho} \text{ArcTan}\left[\frac{2 \rho b \text{Cos}[\varphi]}{b^2 - \rho^2}\right]\right) /. \rho \rightarrow b // \text{Simplify}$$

$$\frac{\text{Sec}[\varphi]}{b}$$

So the charge density goes as the inverse of the cosine of the angle – it piles up right around the gaps as is clear intuitively.

`Plot[Sec[φ], {φ, 0, 2 π}]`



- Graphics -

You may find the relation $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan[x]$ useful.

■ Problem 22 Jackson3ed, problem 3.10.

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero while the cylindrical surface is made of two equal half cylinders, one at potential V and the other at $-V$. Take the potential on the cylindrical surface as

$$V(\varphi) = \begin{cases} V & -\pi/2 < \varphi < \pi/2 \\ -V & \pi/2 < \varphi < 3\pi/2 \end{cases}$$

a) Find the potential inside the cylinder.

b) Assuming $L \gg b$, consider the potential at $z = \frac{L}{2}$ as a function of ρ and φ and compare with the two dimensional problem 2.13.

Solution Problem 22

a) To get zero on both ends we need sinusoidal solutions in z , $\sin(n\pi \frac{z}{L})$ with $n' = 1, 2, 3, \dots$ and since the whole circle is included, continuity at $\varphi = 0$ implies that the φ dependence is sinusoidal. The symmetry under $\varphi \rightarrow -\varphi$ implies we should choose $\cos m \varphi$ where $m = 0, 1, 2, \dots$. The radial dependence is given by the modified Bessel function which is regular at the origin,

$I_m(n\pi \frac{\rho}{L})$. Check that this product is a solution of Laplace's equation:

$$\text{In[146]:= } \mathbf{f} = \mathbf{BesselI}\left[\mathbf{m}, \mathbf{n} \pi \frac{\rho}{L}\right] \mathbf{Cos}\left[\mathbf{m} \varphi\right] \mathbf{Sin}\left[\mathbf{n} \pi \frac{z}{L}\right];$$

$$\text{In[149]:= } \frac{1}{\rho} \partial_{\rho} (\rho \partial_{\rho} \mathbf{f}) + \frac{1}{\rho^2} \partial_{\varphi, \varphi} \mathbf{f} + \partial_{z, z} \mathbf{f} // \mathbf{FullSimplify}$$

Out[149]= 0

It works, so take a linear combination as the solution:

$$\Phi(\rho, \varphi, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} I_m(n\pi \frac{\rho}{L}) \cos m\varphi \sin(n\pi \frac{z}{L})$$

where A_{nm} is chosen to match the boundary condition on the cylindrical surface:

$$V(\varphi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} I_m(n\pi \frac{\rho}{L}) \cos m\varphi \sin(n\pi \frac{z}{L}) = \begin{pmatrix} V & -\pi/2 < \varphi < \pi/2 \\ -V & \pi/2 < \varphi < 3\pi/2 \end{pmatrix} \text{ for } 0 < z < L \text{ and } 0 < \varphi < 2\pi.$$

First dissolve the sum on n by multiplying both sides by $\sin(n'\pi \frac{z}{L})$, $n' = 1, 2, 3 \dots$ and then integrating on z from 0 to L . This step gives

$$\int_0^L \sin(n'\pi \frac{z}{L}) V(\varphi) dz = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} I_m(n\pi \frac{\rho}{L}) \cos m\varphi \int_0^L \sin(n'\pi \frac{z}{L}) \sin(n\pi \frac{z}{L}) dz,$$

$$V(\varphi) \frac{L}{n'\pi} (\cos n'\pi - 1) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} I_m(n\pi \frac{\rho}{L}) \cos m\varphi \delta_{nn'} \frac{L}{2} = \frac{L}{2} \sum_{m=0}^{\infty} A_{n' m} I_m(n'\pi \frac{\rho}{L}) \cos m\varphi,$$

Next, dissolve the sum over m similarly.

$$\frac{L}{n'\pi} (\cos n'\pi - 1) \int_0^{2\pi} \cos m'\varphi V(\varphi) d\varphi =$$

$$\frac{L}{2} \sum_{m=0}^{\infty} A_{n' m} I_m(n'\pi \frac{\rho}{L}) \int_0^{2\pi} \cos m'\varphi \cos m\varphi d\varphi = \pi \frac{L}{2} A_{n' m'} I_{m'}(n'\pi \frac{\rho}{L})$$

Next do the integral remaining.

$$\int_0^{2\pi} \cos m'\varphi V(\varphi) d\varphi = V(\int_{-\pi/2}^{\pi/2} \cos m'\varphi d\varphi - \int_{\pi/2}^{3\pi/2} \cos m'\varphi d\varphi) = -\frac{V}{m'} 4 \sin m' \frac{\pi}{2}$$

So evaluate the two integrals numerically.

```
In[150]:= {#, Cos[# π] - 1, Sin[#  $\frac{\pi}{2}$ ]} & /@ Range[10] // TableForm
```

```
Out[150]//TableForm=
```

1	-2	1
2	0	0
3	-2	-1
4	0	0
5	-2	1
6	0	0
7	-2	-1
8	0	0
9	-2	1
10	0	0

This shows that only odd values of n' and m' contribute and so letting $n' = 2n + 1$ and $m' = 2m + 1$, with both n and m running from 0 to ∞ , and suitably relabeling the constants we get:

$$A_{nm} = (-1)^m \left(\pi \frac{L}{2} I_{2m+1} \left((2n+1) \pi \frac{a}{L} \right) \right)^{-1} \frac{8L}{(2n+1)\pi} \frac{V}{m'} = (-1)^m \frac{16V}{(2n+1)(2m+1)\pi^2} \frac{1}{I_{2m+1} \left((2n+1) \pi \frac{a}{L} \right)}.$$

So, the potential inside a cylinder as given in Jackson problem 3.10 is

$$\Phi(\rho, \varphi, z) = \frac{16V}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2n+1)(2m+1)} \sin((2n+1) \pi \frac{z}{L}) \cos((2m+1) \varphi) \frac{I_{2m+1} \left((2n+1) \pi \frac{\rho}{L} \right)}{I_{2m+1} \left((2n+1) \pi \frac{b}{L} \right)}.$$

b) Let $z = \frac{L}{2}$ and then take the limit as L goes to infinity.

$$\begin{aligned} \Phi(\rho, \varphi, \frac{L}{2}) &= \frac{16V}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2n+1)(2m+1)} \sin((2n+1) \frac{\pi}{2}) \cos((2m+1) \varphi) \frac{I_{2m+1} \left((2n+1) \pi \frac{\rho}{L} \right)}{I_{2m+1} \left((2n+1) \pi \frac{b}{L} \right)} \\ &= \frac{16V}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^n}{(2n+1)(2m+1)} \cos((2m+1) \varphi) \frac{I_{2m+1} \left((2n+1) \pi \frac{\rho}{L} \right)}{I_{2m+1} \left((2n+1) \pi \frac{b}{L} \right)} \end{aligned}$$

Now take the limit as $L \rightarrow \infty$ using $I_k(x) \rightarrow \frac{1}{k!} \left(\frac{x}{2} \right)^k$. Ignoring the point that this approximation does not rigorously work over the entire sum, you get

$$\Phi(\rho, \varphi, \frac{L}{2}) = \frac{16V}{\pi^2} \operatorname{Re} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^n}{(2n+1)(2m+1)} e^{i(2m+1) \varphi} \left(\frac{\rho}{b} \right)^{2m+1}.$$

Next for the sums over n and m use

$$\begin{aligned} &\text{Series} \left[-\frac{1}{2i} \operatorname{Log} \left[\frac{1-i x}{1+i x} \right], \{x, 0, 9\} \right] \\ &x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + O[x]^{10} \end{aligned}$$

The n sum gives $-\frac{1}{2i} \log \left(\frac{1-i}{1+i} \right)$ and the m sum gives $-\frac{1}{2i} \log \left(\frac{1-i e^{i\varphi} \frac{\rho}{b}}{1+i e^{i\varphi} \frac{\rho}{b}} \right)$.

$$\text{Simplify}\left[\text{ComplexExpand}\left[\frac{16V}{\pi^2} \text{Re}\left[-\frac{1}{2i} \text{Log}\left[\frac{1-i}{1+i}\right]\right]\left(-\frac{1}{2i} \text{Log}\left[\text{ComplexExpand}\left[\frac{1-i e^{i\varphi} \frac{\rho}{b}}{1+i e^{i\varphi} \frac{\rho}{b}}\right]\right]\right)\right], \{\rho > 0, b > 0, \varphi \in \text{Reals}\}\right]$$

$$-\frac{2V \text{Arg}[b^2 - \rho^2 - 2i b \rho \text{Cos}[\varphi]]}{\pi}$$

So, in the limit of $L \rightarrow \infty$ the potential near the central region of the cylinder is

$$\lim_{L \rightarrow \infty} \Phi(\rho, \varphi, \frac{L}{2}) = \frac{2V}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2 - \rho^2} \cos \varphi\right),$$

in agreement with the result of Jackson problem 2.13.

Don't forget about the Bessel functions that are like exponentials with real argument which are used in this solution. They are called the *modified* Bessel functions.