The Radial Oscillatory Solutions in 2D

Laplace's equation in two dimensional polar coordinates is

$$\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2\Phi}{\partial\varphi^2}=0.$$

Choosing $\Phi = R(\rho)\,F(\varphi)$ we easily get $\frac{d^2F}{d\varphi^2} = -k^2\,F$ and $\rho\,\frac{d}{d\rho}\left(\rho\,\frac{dR}{d\rho}\right) - k^2\,R = 0$. If k is taken to be real, then the solutions are sinusoids in φ and power functions (or if $k = 0$, the logarithmic function) in ρ. These are suitable for cases in which boundaries of constant φ are grounded and arbitrary functions of φ are specified on boundaries of given radius. But if boundaries at fixed values of ρ are at zero potential and potentials are given on boundaries of fixed φ, then you can get a solution using these functions only if you first find the Green function in terms of them, and then use Green's theorem to get the potential. This is always a feasible approach, but it would be more direct to use solutions that are oscillatory in the radial direction. These are easily gotten by replacing k^2 by $-k^2$, with real k, as the separation variable. Then the solutions become exponentials in φ (or, if more convenient, hyperbolic sines and cosines) and sinusoids in $\log\rho$. In particular, the radial functions can be written as $\rho^{\pm i\,k}$, or as $e^{\pm i\,k\log\rho}$, or as $\sin(k\log\rho)$ and $\cos(k\log\rho)$. It is perfectly straightforward to deal with the case that the potential is set to zero on $\varphi = 0$, $\rho = \rho_0$, and $\rho = \epsilon > 0$ and has a specified functional form $f(\rho)$ on the surface $\varphi = \varphi_0$. The general case can be obtained by superimposing suitable variations on this example.

Take the form

$$\Phi(\rho,\,\varphi) = \sum_{n=1}^{\infty} A_n \sinh k_n\varphi\;\sin\left(k_n \log\frac{\rho}{\rho_0}\right)$$

which is constructed to satisfy the zero values at $\varphi = 0$ and $\rho = \rho_0$. Next choose the k_n so that we get zero value at $\rho = \epsilon$. This is easy; just take $k_n = \frac{n\pi}{\log\frac{\epsilon}{\rho_0}}$ so we have

$$\Phi(\rho,\,\varphi) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\,\pi\,\varphi}{\log\frac{\epsilon}{\rho_0}}\right)\sin\left(n\,\pi\,\frac{\log\frac{\rho}{\rho_0}}{\log\frac{\epsilon}{\rho_0}}\right)$$

Finally select the A_n so that

$$f(\rho) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\,\pi\,\varphi_0}{\log\frac{\epsilon}{\rho_0}}\right)\sin\left(n\,\pi\,\frac{\log\frac{\rho}{\rho_0}}{\log\frac{\epsilon}{\rho_0}}\right).$$

The way to do this is just to multiply both sides by $\sin\left(m\,\pi\,\frac{\log\frac{\rho}{\rho_0}}{\log\frac{\epsilon}{\rho_0}}\right)d\left(\log\frac{\rho}{\rho_0}\right)$ and then integrate over ρ from ϵ to ρ_0, or expressing this in terms of the log, integrate over $\log\frac{\rho}{\rho_0}$ from $\log\frac{\epsilon}{\rho_0}$ to 0. Since when $n \neq m$ the integral is over an integral number of half cycles of the sinusoid, the integral vanishes. When $n = m$ the integral is half the range of the integration. Thus we get

$$A_m \sinh\left(\frac{m\,\pi\,\varphi_0}{\log\frac{\epsilon}{\rho_0}}\right)\frac{1}{2}\log\frac{\epsilon}{\rho_0} = \int_{\log\frac{\epsilon}{\rho_0}}^{0} f(\rho)\sin\left(m\,\pi\,\frac{\log\frac{\rho}{\rho_0}}{\log\frac{\epsilon}{\rho_0}}\right)d\left(\log\frac{\rho}{\rho_0}\right) = \int_{\log\frac{\epsilon}{\rho_0}}^{0} f(\rho_0\,e^{\xi})\sin\left(m\,\pi\,\frac{\xi}{\log\frac{\epsilon}{\rho_0}}\right)d\xi$$

or solving for A_n you get

$$A_n = \frac{2}{\sinh\left(\frac{n\pi\varphi_0}{\log\frac{\epsilon}{\rho_0}}\right)\log\frac{\epsilon}{\rho_0}} \int_{\log\frac{\epsilon}{\rho_0}}^{0} f(\rho_0\, e^{\xi}) \sin\left(n\pi\,\frac{\xi}{\log\frac{\epsilon}{\rho_0}}\right) d\xi$$

■ An example

As an example, take the potential to be a constant, V. Then the integral becomes

$$\int_{\log\frac{\epsilon}{\rho_0}}^{0} V \sin\left(n\pi\,\frac{\xi}{\log\frac{\epsilon}{\rho_0}}\right) d\xi = -V\,\frac{\log\frac{\epsilon}{\rho_0}}{n\pi}\cos\left(n\pi\,\frac{\xi}{\log\frac{\epsilon}{\rho_0}}\right)\Bigg|_{\xi=\log\frac{\epsilon}{\rho_0}}^{0} = V\,\frac{\log\frac{\epsilon}{\rho_0}}{n\pi}\,(\cos n\pi\, -\, 1) = V\,\frac{\log\frac{\epsilon}{\rho_0}}{n\pi}\begin{pmatrix} 0 & n\,\text{even} \\ -2 & n\,\text{odd} \end{pmatrix}.$$

So in this case the solution becomes

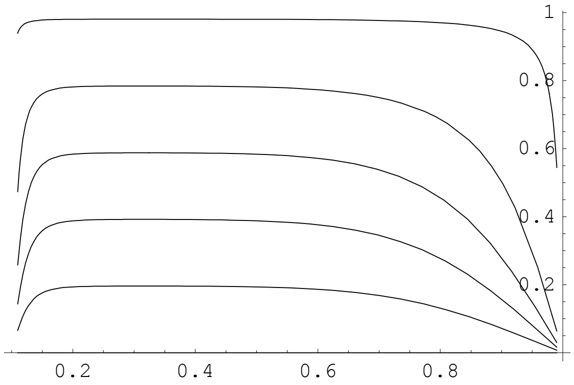
$$\Phi(\rho,\,\varphi) = \frac{V}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\,\frac{4}{\sinh\left(\frac{(2n+1)\pi\varphi_0}{\log\frac{\rho_0}{\epsilon}}\right)}\,\sinh\left(\frac{(2n+1)\pi\varphi}{\log\frac{\rho_0}{\epsilon}}\right)\sin\left((2n+1)\pi\,\frac{\log\frac{\rho_0}{\rho}}{\log\frac{\rho_0}{\epsilon}}\right)$$

The following plot shows the potential as a function of radius at several different values of the angle φ. It is clear that at the "upper corners" of the space this series does not give a good representation of the potential, but, of course, at these points the potential is formally discontinuous and it is hardly surprising that the series will have convergence problems.

$$\texttt{Plot}\Big[\texttt{Evaluate}\Big[\texttt{With}\Big[\{\varphi0 = 25.5\,\texttt{Degree},\ \rho0 = 1.,\ \epsilon = .1,\ \varphi = \#\,\texttt{Degree}\},$$

$$\sum_{n=0}^{1000}\frac{4\,\texttt{Sinh}\left[\frac{(2\,n+1)\,\pi\,\varphi}{\texttt{Log}\left[\frac{\rho0}{\epsilon}\right]}\right]\texttt{Sin}\left[\frac{(2\,n+1)\,\pi\,\texttt{Log}\left[\frac{\rho0}{\rho}\right]}{\texttt{Log}\left[\frac{\rho0}{\epsilon}\right]}\right]}{(2\,n+1)\,\pi\,\texttt{Sinh}\left[\frac{(2\,n+1)\,\pi\,\varphi0}{\texttt{Log}\left[\frac{\rho0}{\epsilon}\right]}\right]}\Big]\ \&\ /@$$

$$\texttt{Table}[\varphi,\ \{\varphi,\ 0,\ 25,\ 5\}]\Big],\ \{\rho,\ .11,\ .99\},\ \texttt{PlotRange} \to \texttt{All}\Big]$$



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- Graphics -
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