Hamiltonian Action Principle for Fields
Part 1

Action Principle for Classical Mechanics

As you know from the a study of the classical mechanics of point masses and rigid bodies, the equations of motion can be derived by appealing to a general principle discovered by Hamilton. It is called the Principle of Least Action (actually it should be called the Principle of Stationary Action) and states that, for a mechanical system described by generalized coordinates $q_i, \; i = 1, n$ with action

$$A = \int_{t_1}^{t_2} L\left(\frac{dq_i(t)}{dt}, q_i(t), t\right) dt$$

where L, the Lagrangian, is $L = T - V$, the motion $q_i(t)$ of the system, with $q_i(t_1)$ and $q_i(t_2)$ specified, has the property that A is left unchanged to first order in $\delta q_i(t)$ if $q_i(t)$ is replaced by $q_i(t) + \delta q_i(t)$ so long as $\delta q_i(t_1) = \delta q_i(t_2) = 0$. In the Lagrangian, T is the kinetic energy of the system and V, the potential energy. The demonstration is very simple, simply requiring the expansion of $L\left(\frac{dq_i(t)}{dt} + \frac{d\delta q_i}{dt}, q_i(t) + \delta q_i(t), t\right)$ to first order in $\delta q_i(t)$. The result is

$$A + \delta A = \int_{t_1}^{t_2} L\left(\frac{dq_i(t)}{dt} + \frac{d\delta q_i}{dt}, q_i(t) + \delta q_i(t), t\right) dt$$
$$= \int_{t_1}^{t_2} \left(L\left(\frac{dq_i(t)}{dt}, q_i(t), t\right) + \frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \frac{d\delta q_i}{dt} + \frac{\partial L}{\partial q_i(t)} \delta q_i(t) \right) dt.$$

The main trick is to complete the derivative with respect to t in the second term, making the necessary correction to get

$$\delta A = \int_{t_1}^{t_2} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \delta q_i(t) \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \right) \delta q_i(t) + \frac{\partial L}{\partial q_i(t)} \delta q_i(t) \right) dt.$$

The complete derivative with respect to time can be integrated to yield

$$\left(\frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \delta q_i \right) \Bigg|_{t=t_1}^{t=t_2}.$$

This is zero because of the conditions that the functions δq_i vanish on the boundaries of the integration, $t = t_1, t_2$. This then gives

$$\delta A = \int_{t_1}^{t_2} \left(-\frac{d}{dt} \left(\frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \right) + \frac{\partial L}{\partial q_i(t)} \right) \delta q_i(t) \, dt.$$

This can be independent of any $\delta q_i(t)$ satisfying the boundary conditions if and only if the coefficient of $\delta q_i(t)$ vanishes for all t in the open interval (t_1, t_2). We impose continuity to extend this to the closed interval, and so we get Lagrange's celebrated equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial\left(\frac{dq_i(t)}{dt}\right)} \right) - \frac{\partial L}{\partial q_i(t)} = 0 \text{ for } i = 1, n \text{ in } t \in [t_1, t_2].$$

Action Principle for a Field Theory

We can take a field to be a set of physical functions defined over space-time, or more generally, a set of several independent variables. Classical mechanics has just a single independent variable, time, but in electrostatics, for example, there are three independent variables defining a point in 3D space. Let us suppose a set of fields A_i, $i = 1, n$ defined over a set of in dependent variables q_j, $j = 1, m$.

Assume a *Lagrangian density,* $\mathcal{L}\left(\frac{\partial A_i}{\partial q_j}, A_i, q_j\right)$ and use it to define an action in a volume V in the space $(q_1, q_2, \ldots, q_m)$,

$$\mathcal{A} = \int_V \mathcal{L}\left(\frac{\partial A_i}{\partial q_j}, A_i, q_j\right) dq_1\, dq_2 \ldots dq_m \qquad \text{(eq 1).}$$

Then for an infinitesimal variation $\delta A_i(q_j)$ which vanishes on ∂V, the surface of the volume V, it is easy to see that

$$\mathcal{A} + \delta\mathcal{A} = \int_V \mathcal{L}\left(\frac{\partial A_i}{\partial q_j} + \frac{\partial\, \delta A_i}{\partial q_j}, A_i + \delta A_i, q_j\right) dq_1\, dq_2 \ldots dq_m$$
$$= \int_V \left(\mathcal{L}\left(\frac{\partial A_i}{\partial q_j} + \frac{\partial\, \delta A_i}{\partial q_j}, A_i + \delta A_i, q_j\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial q_j}\right)} \frac{\partial\, \delta A_i}{\partial q_j} + \frac{\partial \mathcal{L}}{\partial A_i} \delta A_i\right) dq_1\, dq_2 \ldots dq_m.$$

Then, completing the derivative yields

$$\delta\mathcal{A} = \int_V \left(\frac{\partial}{\partial q_j}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial q_j}\right)} \delta A_i\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial q_j}\right)}\right)\delta A_i + \frac{\partial \mathcal{L}}{\partial A_i} \delta A_i\right) dq_1\, dq_2 \ldots dq_m.$$

Now apply Gauss's theorem, the multidimensional version of the integral of a total derivative, and use the vanishing of $\delta A_i(q_j)$ on ∂V to get

$$\delta\mathcal{A} = \int_V \left(-\frac{\partial}{\partial q_j}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial q_j}\right)}\right) + \frac{\partial \mathcal{L}}{\partial A_i}\right)\delta A_i\, dq_1\, dq_2 \ldots dq_m.$$

Again, if this is to vanish for any δA_i which vanishes on ∂V we get Lagrange's equations for the fields:

$$\frac{\partial}{\partial q_j}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial q_j}\right)}\right) - \frac{\partial \mathcal{L}}{\partial A_i} = 0, \quad i = 1\, m \text{ and the sum on } j \text{ runs over } j = 1, m \qquad \text{(eq 2).}$$

Strictly, Lagrange's equations are not required to hold on ∂V itself, only inside V, but requiring continuity allows us to extend them.

Action Principle for Electrostatics

It is not obvious what we ought to take as the field to get a Lagrangian formulation of electrostatics, but, taking our lead from classical mechanics in which the *potential energy* played the crucial role rather than the force, we choose the potential $\Phi(\vec{r})$ and take ordinary space (x, y, z) as the independent coordinates. What should we try for the Lagrangian? Again taking the lead from mechanics, we choose a linear combination of two expressions for energy: a "kinetic" term, $\left(\vec{\nabla}\Phi\right)^2$, and a "potential" term, $\rho\,\Phi$. So with these hints from mechanics we consider

$$\mathcal{L} = a\left(\vec{\nabla}\Phi\right)^2 + b\,\rho\,\Phi$$

where a and b are constants. Of course, for our purposes, the overall numerical value of the Lagrangian density is irrelevant so only the ratio of a and b is of interest, and that can be taken as we choose depending only on our choice of units in which to measure charge. Thus to get the normal SI units, it turns out that we should take

$$\boxed{\mathcal{L} = \frac{1}{2}\,\varepsilon_0\left(\vec{\nabla}\Phi\right)^2 - \rho\,\Phi = \frac{1}{2}\,\varepsilon_0\left(\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right) - \rho\,\Phi}$$

where Cartesian coordinates are used. For the first term in Lagrange's equation for the field Φ we need

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial x}\right)} = \varepsilon_0\,\frac{\partial\Phi}{\partial x},\quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial y}\right)} = \varepsilon_0\,\frac{\partial\Phi}{\partial y},\quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial z}\right)} = \varepsilon_0\,\frac{\partial\Phi}{\partial z},$$

and for the second term,

$$\frac{\partial\mathcal{L}}{\partial\Phi} = -\rho.$$

So we get Poisson's equation with this Lagrangian:

$$\varepsilon_0\,\frac{\partial^2\Phi}{\partial x^2} + \varepsilon_0\,\frac{\partial^2\Phi}{\partial y^2} + \varepsilon_0\,\frac{\partial^2\Phi}{\partial z^2} + \rho = 0 \quad \Rightarrow \nabla^2\,\Phi = -\frac{\rho}{\varepsilon_0}.$$

Electrostatics in Curvilinear Orthogonal Coordinates

■ Curvilinear Coordinates

One nice use of the Lagrangian is that it gives us a straight forward way to formulate the field equations in coordinate systems other than Cartesian. The most useful such systems (q_1, q_2, q_3) are curvilinear and orthogonal. "Curvilinear" means that each of (x, y, z) can be expressed as a non-linear function of the three arguments (q_1, q_2, q_3) and that as each of (q_1, q_2, q_3) vary over some specified range, all points in (x, y, z) are covered just once except for possible singular points which are in general multicovered. "Orthogonal" means that, at a point in space, the differential vectors $d\vec{r}_{q_j}$ generated by dq_j satisfy $d\vec{r}_{q_j} \cdot d\vec{r}_{q_i} = 0$ if $i \neq j$. Thus at each point in space there is a local cartesian system defined by the three infinitesimal vectors $d\vec{r}_{q_j}$. Unit vectors $\vec{u}_{q_j}$ along the directions of the three differential vectors $d\vec{r}_{q_j}$ are often more convenient.

More explicitly, expressed as components in the underlying cartesian system, we have the vector

$$d\vec{r}_{q_j} = \left(\frac{\partial x(q_1, q_2, q_3)}{\partial q_j}, \frac{\partial y(q_1, q_2, q_3)}{\partial q_j}, \frac{\partial z(q_1, q_2, q_3)}{\partial q_j}\right) dq_j \quad \text{(no sum over } j\text{)}.$$

It is very useful to define a set of functions called scale factors

$$h_j(q_1, q_2, q_3) = \sqrt{\left(\frac{\partial x(q_1, q_2, q_3)}{\partial q_j}\right)^2 + \left(\frac{\partial y(q_1, q_2, q_3)}{\partial q_j}\right)^2 + \left(\frac{\partial z(q_1, q_2, q_3)}{\partial q_j}\right)^2}$$

so that $d\vec{r}_{q_j} = h_j \vec{u}_{q_j} dq_j$ (no sum over j) where as above the $\vec{u}_{q_j}$ are unit vectors. It is customary to order the unit vectors by the right hand rule, namely that

$$\vec{u}_{q_1} \times \vec{u}_{q_2} = \vec{u}_{q_3}.$$

Further note that, as is the usual convention, the positive square root is to be used since there is no prepended minus sign. It is also useful to notice that the components of the vectors $d\vec{r}_{q_j}$ are used to construct the Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{pmatrix} = \begin{pmatrix} h_1 (\vec{u}_{q_1})_x & h_2 (\vec{u}_{q_2})_x & h_3 (\vec{u}_{q_3})_x \\ h_1 (\vec{u}_{q_1})_y & h_2 (\vec{u}_{q_2})_y & h_3 (\vec{u}_{q_3})_y \\ h_1 (\vec{u}_{q_1})_z & h_2 (\vec{u}_{q_2})_z & h_3 (\vec{u}_{q_3})_z \end{pmatrix}$$

$$= \begin{pmatrix} (\vec{u}_{q_1})_x & (\vec{u}_{q_2})_x & (\vec{u}_{q_3})_x \\ (\vec{u}_{q_1})_y & (\vec{u}_{q_2})_y & (\vec{u}_{q_3})_y \\ (\vec{u}_{q_1})_z & (\vec{u}_{q_2})_z & (\vec{u}_{q_3})_z \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = \mathcal{O}\mathcal{H}.$$

The absolute value of the determinate of $\mathcal{J}$ is the Jacobian of the transformation from (x, y, z) to (q_1, q_2, q_3). This gives us the law for expressing a volume element in the (q_1, q_2, q_3) system:

$$dx\, dy\, dz = |\det \mathcal{J}|\, dq_1\, dq_2\, dq_3.$$

Because the three $\vec{u}$ are orthogonal unit vectors, the matrix O is orthogonal and has unit determinate with the right-hand rule ordering convention adopted above. So we get the important rule

$$dx\,dy\,dz = h_1\,h_2\,h_3\,dq_1\,dq_2\,dq_3.$$

■ The Gradient in Curvilinear Coordinates

The gradient arising in the electrostatic Lagrangian is defined so that for two points separated by $d\vec{r}$ the change in a scalar function Φ is $d\Phi = \vec{\nabla}\Phi\cdot d\vec{r}$. Thus if the two points are defined by q_j and $q_j + dq_j$, then we can immediately write

$$d\Phi = \frac{\partial\Phi}{\partial q_j}\,dq_j \qquad \text{and} \qquad d\vec{r} = h_1\,\vec{u}_{q_1}\,dq_1 + h_2\,\vec{u}_{q_2}\,dq_2 + h_3\,\vec{u}_{q_3}\,dq_3.$$

So we get

$$\frac{\partial\Phi}{\partial q_1}\,dq_1 + \frac{\partial\Phi}{\partial q_2}\,dq_2 + \frac{\partial\Phi}{\partial q_3}\,dq_3 = \left(\vec{\nabla}\Phi\right)_{q_1} h_1\,dq_1 + \left(\vec{\nabla}\Phi\right)_{q_2} h_2\,dq_2 + \left(\vec{\nabla}\Phi\right)_{q_3} h_3\,dq_3.$$

Finally then we get the important and easily remembered formula for the gradient in curvilinear coordinates:

$$\vec{\nabla}\Phi = \frac{1}{h_1}\,\frac{\partial\Phi}{\partial q_1}\,u_{q_1} + \frac{1}{h_2}\,\frac{\partial\Phi}{\partial q_2}\,u_{q_2} + \frac{1}{h_3}\,\frac{\partial\Phi}{\partial q_3}\,u_{q_3}.$$

This is easy to remember if you just realize that $h_1\,dq_1$ is the geometric distance associated with the coordinate change dq_1 and similarly for the other two directions.

■ The Action in Curvilinear Coordinates

In Cartesian coordinates we have for the action

$$\mathcal{A} = \int\left(\frac{1}{2}\,\varepsilon_0\left(\vec{\nabla}\Phi\right)^2 - \rho\,\Phi\right)dx\,dy\,dz \text{ where} \left(\vec{\nabla}\Phi\right)^2 = \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2$$

Transforming to (q_1, q_2, q_3) produces

$$\mathcal{A} = \int\left(\frac{1}{2}\,\varepsilon_0\left(\vec{\nabla}\Phi\right)^2 - \rho\,\Phi\right)h_1\,h_2\,h_3\,dq_1\,dq_2\,dq_3$$

where

$$\left(\vec{\nabla}\Phi\right)^2 = \left(\frac{1}{h_1}\,\frac{\partial\Phi}{\partial q_1}\right)^2 + \left(\frac{1}{h_2}\,\frac{\partial\Phi}{\partial q_2}\right)^2 + \left(\frac{1}{h_3}\,\frac{\partial\Phi}{\partial q_3}\right)^2.$$

Casting this into the general form of eq 1 we get

$$\mathcal{A} = \int\mathcal{L}\,dq_1\,dq_2\,dq_3$$

where $\mathcal{L} = h_1\, h_2\, h_3 \left(\frac{1}{2}\, \varepsilon_0 \left(\left(\frac{1}{h_1}\, \frac{\partial \Phi}{\partial q_1}\right)^2 + \left(\frac{1}{h_2}\, \frac{\partial \Phi}{\partial q_2}\right)^2 + \left(\frac{1}{h_3}\, \frac{\partial \Phi}{\partial q_3}\right)^2\right) - \rho\, \Phi\right)$.

■ The Electrostatic Field Equation in Curvilinear Coordinates

It is now easy to apply the Lagrange equations eq 2 to get the electrostatic field equation in arbitrary curvilinear orthogonal coordinates.

$$\frac{\partial \mathcal{L}}{\left(\frac{\partial \Phi}{\partial q_1}\right)} = \frac{h_2\, h_3}{h_1}\, \frac{\partial \Phi}{\partial q_1},\ \frac{\partial \mathcal{L}}{\left(\frac{\partial \Phi}{\partial q_2}\right)} = \frac{h_1\, h_3}{h_2}\, \frac{\partial \Phi}{\partial q_2},\ \frac{\partial \mathcal{L}}{\left(\frac{\partial \Phi}{\partial q_3}\right)} = \frac{h_1\, h_2}{h_3}\, \frac{\partial \Phi}{\partial q_3},\ \frac{\partial \mathcal{L}}{\partial \Phi} = -h_1\, h_2\, h_3\, \frac{\rho}{\varepsilon_0}$$

and so the field equation becomes

$$\frac{1}{h_1\, h_2\, h_3} \left(\frac{\partial}{\partial q_1} \left(\frac{h_2\, h_3}{h_1}\, \frac{\partial \Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2} \left(\frac{h_1\, h_3}{h_2}\, \frac{\partial \Phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3} \left(\frac{h_1\, h_2}{h_3}\, \frac{\partial \Phi}{\partial q_3}\right)\right) = -\frac{\rho}{\varepsilon_0}.$$

We can use this relation for $\nabla^2 \Phi = \vec{\nabla} \cdot \vec{\nabla} \Phi$ and the earlier derived for the gradient to get the expression for the divergence in curvilinear orthogonal coordinates. The result is

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1\, h_2\, h_3} \left(\frac{\partial}{\partial q_1}\, (h_2\, h_3\, A_{q_1}) + \frac{\partial}{\partial q_2}\, (h_1\, h_3\, A_{q_2}) + \frac{\partial}{\partial q_3}\, (h_1\, h_2\, A_{q_3})\right).$$

■ Example: Spherical Polar Coordinates

Let the transformation be defined by

$$x = r \cos\varphi \sin\theta$$
$$y = r \sin\varphi \sin\theta$$
$$z = r \cos\theta$$

where $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. The figure associated with this transformation is sufficiently familiar that I will not take the time to make it here. The Jacobian matrix is easy to get:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi \sin\theta & r \cos\varphi \cos\theta & -r \sin\varphi \sin\theta \\ \sin\varphi \sin\theta & r \sin\varphi \cos\theta & r \cos\varphi \sin\theta \\ \cos\theta & -r \sin\theta & 0 \end{pmatrix}$$

From this we get the scale factors: $h_r = 1$, $h_\theta = r$, $h_\varphi = r \sin\theta$. A simple diagram makes these factors easy to derive geometrically. The electrostatic field equation then becomes

$$\frac{1}{r^2 \sin\theta} \left(\frac{\partial}{\partial r} \left(\frac{r^2 \sin\theta}{1}\, \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta} \left(\frac{r \sin\theta}{r}\, \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial \varphi} \left(\frac{r}{r \sin\theta}\, \frac{\partial \Phi}{\partial \varphi}\right)\right) = -\frac{\rho}{\varepsilon_0}$$

$\Rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=-\frac{\rho}{\varepsilon_0}.$$

This is of course is just what you get by using the inside of the back cover of Jackson's book. We can also use it to get the expression for the divergence in spherical coordinates. From above,

$$\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta\,A_r)+\frac{\partial}{\partial\theta}(r\sin\theta\,A_\theta)+\frac{\partial}{\partial\varphi}(r\,A_\varphi)\right)$$
$$=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}.$$

General Considerations for Electrostatics

It is natural to ask if there is any other use for a Lagrangian density in electrostatics than finding Laplacian operators in other coordinate systems — except for this, it just gives us back what we already know to be true. One important use is to find symmetries in the theory; they can often be more evident in the Lagrangian than in the field equations themselves. And in the case that a symmetry depends upon one or more continuous parameters, it is possible to deduce "conservation" laws using a method invented by Emmy Noether. A more philosophic use is to give a top down overview of a field theory and ask what are the minimal theoretic postulates or principles from which it can be formulated. This idea is motivated by the hope that these principles can be extrapolated to other fields to learn new physics. Here I give this overview for source-free electrostatics — a rather empty theory, but of theoretic interest nonetheless..

■ The Source-free Scalar Field

Suppose one poses the question: What is the most general source-free scalar field theory over 3D space that is invariant under spatial rotations and translations, is local, and is linear? The first two conditions simply mean that, without any sources to distinguish one place or direction in space, it is reasonable to seek a fundamental theory that is homogeneous and isotropic — no direction or position in space special. The second condition above is less compelling but it states the prejudice that what is going on here depends only upon the immediate vicinity of here, and not what is going on "far away". And finally linearity is quite special and is included only because we know that many physical phenomena (but by no means all) satisfy linearity. The linearity condition is primarily included in anticipation of the introduction of sources.

■ Translational invariance

This means that the Lagrangian density must not depend directly on position $\vec{r}$ or any functions of it, excepting the field Φ and its derivatives themselves.

■ Rotational invariance

This means that the Lagrangian density must be a scalar. Any vector that appears in it, such as $\frac{\partial\Phi}{\partial x_j}$, must appear in a dot product.

■ Locality

By a **local** theory I mean that the Lagrangian depends only on the field at a spatial point $\vec{r}$ and its first spatial derivatives of Φ there. Intuitively you can think of first derivatives as depending only on the value of the function at "nearest neighbor" points whereas higher derivatives depend upon its values at more remote places or over a wider range. You can think of the finite approximation to the derivatives —

$$\frac{\partial\Phi}{\partial x} \simeq \frac{\Phi(x+\Delta x,y,z)-\Phi(x,y,z)}{\Delta x}.$$

It depends on values of the field in only a small region Δx whereas

$$\frac{\partial^2\Phi}{\partial x^2} \simeq \frac{\Phi(x+2\,\Delta x,y,z)-2\,\Phi(x+\Delta x,y,z)+\Phi(x,y,z)}{(\Delta x)^2}$$

or

$$\frac{\partial^2\Phi}{\partial x^2} \simeq \frac{\Phi(x+\Delta x,y,z)-2\,\Phi(x,y,z)+\Phi(x-\Delta x,y,z)}{(\Delta x)^2}$$

depends on the values in a larger range, $2\,\Delta x$. Of course, these intuitive notions of "nearness" or size of range are not precise mathematically; let's just let local mean dependence on value and first derivatives.

■ Linearity

This condition simply means that Lagrangian density must be quadratic in Φ and its derivatives (no linear terms, or of order higher than quadratic) so that the field equations themselves are linear in the field.

Now it is easy to write the most general Lagrangian for a scalar field Φ that satisfies this set of conditions. It is, in Cartesian coordinates,

$$\mathcal{L}\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}, \Phi\right) = a\,\frac{\partial\Phi}{\partial r_j}\,\frac{\partial\Phi}{\partial r_j} + b\,\Phi^2$$

where a and b are constants. The resulting free-space field equation is

$$\frac{\partial}{\partial r_j}\left(2\,a\,\frac{\partial\Phi}{\partial r_j}\right) - 2\,b\,\Phi = 0$$

Clearly, only the ratio of a and b is relevant and, to avoid a completely vacuous theory, we ought not let $a = 0$! The overall size of the Lagrangian is irrelevant so with no loss in generality we can for convenience set $a = \frac{1}{2}$ and $b = \frac{1}{2\lambda^2}$. Note that in principle λ may be pure real or pure imaginary corresponding to two different classes of theory; it has been unsaid to now, but I will restrict the theory to real valued functions so I don't consider general complex numbers as possible constants. Further, I will assume that λ is real since I am only doing an example. So with the above principles, we get for a source-free scalar field theory

$$\mathcal{L}\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}, \Phi\right) = \frac{1}{2}\,\frac{\partial\Phi}{\partial r_j}\,\frac{\partial\Phi}{\partial r_j} + \frac{1}{2\lambda^2}\,\Phi^2 = \frac{1}{2}\left(\left(\vec{\nabla}\Phi\right)^2 + \frac{1}{\lambda^2}\,\Phi^2\right)$$

implying the source-free field equation

$$\nabla^2\Phi - \frac{1}{\lambda^2}\,\Phi = 0$$

which is not electrostatics! The term linear in Φ is called the "mass term" from how it appears in quantum mechanics.

■ The Yukawa potential

It is interesting to add a unit point source at the origin, $\delta^3(\vec{r})$, and see what this field equation produces. Consider

$$\nabla^2 Y - \frac{1}{\lambda^2}\, Y = -\delta^3(\vec{r}),$$

(Y as in Yukawa). In order to produce the 3D delta function, it is natural to make the substitution

$$Y = \frac{f(r)}{r}$$

where by spherical symmetry we can assume a function f of the radial distance only. Then, from above in spherical coordinates, the field equation becomes

$$\begin{aligned}
&\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\left(\frac{f}{r}\right)\right) - \frac{1}{\lambda^2}\frac{f(r)}{r} \\
&= \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{1}{r}\frac{df}{dr} + f\frac{d}{dr}\left(\frac{1}{r}\right)\right)\right) - \frac{1}{\lambda^2}\frac{f(r)}{r} \\
&= \frac{1}{r^2}\frac{d}{dr}\left(r\frac{df}{dr} + f\, r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right) - \frac{1}{\lambda^2}\frac{f(r)}{r} \\
&= \frac{1}{r^2}\frac{df}{dr} + \frac{1}{r}\frac{d^2 f}{dr^2} - \frac{1}{r^2}\frac{df}{dr} + f\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right) - \frac{1}{\lambda^2}\frac{f(r)}{r} \\
&= \frac{1}{r}\frac{d^2 f}{dr^2} - f(0)\,\delta^3(\vec{r}) - \frac{1}{\lambda^2}\frac{f(r)}{r} \\
&= -\delta^3(\vec{r}).
\end{aligned}$$

Therefore, $f(0) = 1$ and $\frac{d^2 f}{dr^2} - \frac{1}{\lambda^2} f(r) = 0$.

Finally then we get $f = A\, e^{r/\lambda} + B\, e^{-r/\lambda}$, with $A + B = 1$. the solution with exponentially growing values as $r \to \infty$ is unphysical so we finally get for the field due to a point source in this theory

$$Y(r) = \frac{e^{-r/\lambda}}{r}$$

which is the Yukawa potential; it is said to have a limited determined by the constant λ.

■ The Gauge Condition to get Electrostatics

It would be interesting to find a **principle** that forces the "mass term" to be zero, and, in fact, there is one. In general, it is called a gauge symmetry which for our purposes is that the field equations are unchanged when something is added to the field. For electrostatics the symmetry is that the field equation is postulated to be invariant if the potential is replaced by itself plus a constant. This is just the familiar property of an electric potential that you can add a constant to it and no physics is changed.

For the Lagrangian we obtained assuming the four conditions already used we obtained $\frac{1}{2}\left(\left(\vec{\nabla}\Phi\right)^2 + \frac{1}{\lambda^2}\Phi^2\right)$ for the Lagrangian. If we replace Φ by $\Phi + a$ where a is a constant, we get $\frac{1}{2}\left(\left(\vec{\nabla}\Phi\right)^2 + \frac{1}{\lambda^2}\Phi^2 + \frac{2}{\lambda^2}a\,\Phi + \frac{a^2}{\lambda^2}\right)$ which produces a different field equation unless $\frac{1}{\lambda^2} = 0$. Therefore we get ordinary electrostatics in a source free space if we invoke a new fundamental symmetry, namely a suitable gauge symmetry. This is of importance since every fundamental physical field we know about has a gauge symmetry and in fact they were historically key elements in formulating the theories of the electroweak and QCD interactions. We see then that source-free electrostatics is local, homogeneous, isotropic, linear, and gauge invariant; I think it is the simplest field theory.

■ Adding Sources

If we allow homogeneity and isotropy to be violated by putting sources ar distinct places and orientations in space, but maintain the other conditions with the understanding that linearity implies that if the field produced by a sum of sources is the sum of the fields produced by each then the most general Lagrangian in a scalar field (with constants adjusted to accommodate SI units in E&M) is

$$\frac{1}{2}\,\varepsilon_0\left(\vec{\nabla}\Phi\right)^2 - \vec{f}\cdot\vec{\nabla}\Phi - \rho\,\Phi$$

where f and ρ are functions of position. This Lagrangian produces the field equation

$$\nabla^2\Phi = -\frac{\rho+\vec{\nabla}\cdot\vec{f}}{\varepsilon_0}$$

It is clear that no loss of generality is incurred by setting $f = 0$ since this just involves a redefinition of the function ρ. Thus in electrostatics, we have the simplest linear, local, gauge invariant scalar field theory with a source.

■ Lack of Uniqueness of the Lagrangian Density

The physics of a field is given by the field equations so that two Lagrangian densities that produce the same field equations are equivalent and either may be used at one's convenience. Usually, the algebraically simplest form is chosen and we can take $\mathcal{L}' \to \mathcal{L}$ when both $\mathcal{L}'$ and $\mathcal{L}$ produce the same field equations and $\mathcal{L}$ is the standard form, simplest in some sense. Thus for example both $\mathcal{L}$ and $a\,\mathcal{L}$ produce the same field equations where a is an arbitrary constant, and so we can just set a to unity. Similarly, both $\mathcal{L}$ and $\mathcal{L} + f(\vec{r})$ where $f(\vec{r})$ is a function independent of Φ yield the same field equations and so f may be dropped. Another very important case is that $\mathcal{L}' = \mathcal{L} + \frac{\partial F_j(\Phi,\vec{r})}{\partial x_j}$ where $F_j(\Phi, \vec{r})$ is an arbitrary vector function depending on Φ and {x,y,z}, but not $\frac{\partial\Phi}{\partial x_j}$. This additive term is a divergence and so, by Gauss's law, $\int_V \frac{\partial F_j}{\partial x_j}\, dx\, dy\, dz = \int_{\partial V} F_j\, dA_j$, which is just an additive constant to the action and does not affect the field equations. You can see this directly from Lagrange's equations:

$$\mathcal{L}' = \mathcal{L} + \frac{\partial F_j(\Phi,\vec{r})}{\partial x_j} = \mathcal{L} + \frac{\partial F_j(\Phi,\vec{r})}{\partial \Phi}\,\frac{\partial\Phi}{\partial x_j} + \frac{\partial F_j(\Phi,\vec{r})}{\partial x_j}$$

and so the equation for the field is

$$\frac{\partial}{x_j}\left(\frac{\partial\mathcal{L}'}{\partial\left(\frac{\partial\Phi}{\partial x_j}\right)}\right) - \frac{\partial\mathcal{L}'}{\partial\Phi} = \frac{\partial}{x_j}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial x_j}\right)} + \frac{\partial F_j(\Phi,\vec{r})}{\partial\Phi}\right) - \left(\frac{\partial\mathcal{L}}{\partial\Phi} + \frac{\partial^2 F_j(\Phi,\vec{r})}{\partial\Phi^2}\,\frac{\partial\Phi}{\partial x_j} + \frac{\partial^2 F_j(\Phi,\vec{r})}{\partial\Phi\,\partial x_j}\right) = 0.$$

But

$$\frac{\partial}{x_j}\left(\frac{\partial F_j(\Phi,\vec{r})}{\partial\Phi}\right) = \frac{\partial^2 F_j(\Phi,\vec{r})}{\partial\Phi^2}\,\frac{\partial\Phi}{\partial x_j} + \frac{\partial^2 F_j(\Phi,\vec{r})}{\partial\Phi\,\partial x_j}$$

which exactly cancels the last two terms in the field equation above. This shows that both $\mathcal{L}'$ and $\mathcal{L}$ produce exactly the same equations of motion and so are equivalent. The Lagrangian is not unique. This fact is important is studying the symmetries of a theory since if a transformed Lagrangian is the original Lagrangian plus a divergence as used above, the divergence term may be dropped and the two Lagrangians are physically equivalent.