# The Dirac Delta Function in Curvilinear Orthogonal Coordinates

## ■ Non-singular points

When using curvilinear orthogonal coordinates $(q_1, q_2, q_3)$ it is very tempting to write for the three dimensional Dirac delta function

$\delta^3(\vec{r} - \vec{r}\,') = \delta(q_1 - q_1')\,\delta(q_2 - q_2')\,\delta(q_3 - q_3')$ (WRONG in general).

However, this is incorrect except when the coordinates are cartesian. In general, in curvilinear coordinates the correct formula is

$$\delta^3(\vec{r} - \vec{r}\,') = \frac{1}{h_1\,h_2\,h_3}\,\delta(q_1 - q_1')\,\delta(q_2 - q_2')\,\delta(q_3 - q_3') \qquad \text{eq. 1}$$

where $h_i \geq 0$ is the scale factor for $q_i$. This follows from the fundamental property of the delta function, namely, that

$$\int_V \delta^3(\vec{r} - \vec{r}\,')\,d^3 r = 1 \quad \text{if} \quad \vec{r}\,' \in V \qquad \text{eq. 2.}$$

In this equation $d^3 r$ is, as usual, a differential volume of physical space, i.e., $d^3 r = dx\,dy\,dz$ – not parameter space $d^3 r \neq dq_1\,dq_2\,dq_3$. However, a differential volume of physical space can be expressed in terms of a differential volume of parameter space using the Jacobian. For the case of orthogonal coordinates, we must have the important relation given

$$d^3 r = h_1\,h_2\,h_3\,dq_1\,dq_2\,dq_3 \qquad \text{eq. 3.}$$

Consequently, at a non-singular point we must have eq. 1 in the curvilinear coordinates.

## ■ Singular points

Now what about a singular point which is a spatial point that is multiply covered by one or two of the parameters in the curvilinear system. Examples are the origin in 2D polar coordinates (covered by all parameters with $\rho = 0,\ 0 \leq \varphi < 2\pi$), the origin in 3D spherical polar coordinates (covered by all parameters with $\rho = 0,\ 0 \leq \varphi < 2\pi,\ 0 \leq \theta \leq \pi$), and points on the $z$ axis in 3D spherical polar coordinates except the origin (covered by all parameters with $\rho = 0,\ 0 \leq \varphi < 2\pi,\ \theta = 0 \text{ or } \theta = \pi$).

Suppose the coordinate $q_1$takes on values in the range $(q_{11}, q_{12})$, $q_{12} > q_{11}$, and multiply covers the point where the delta function is centered. In this case the representation of $\delta^3(\vec{r} - \vec{r}\,')$ in the curvilinear coordinates will NOT have a factor like $\delta(q_1 - q_1')$ since the point is question has no unique value for $q_1'$ – its multiply covered by the variable $q_1$! Thus in order to satisfy the fundamental eq. 2 we must have

$$\delta^3(\vec{r} - \vec{r}\,') = \frac{1}{\left(\int_{q_{11}}^{q_{12}} h_1\,dq_1\right) h_2\,h_3}\,\delta(q_2 - q_2')\,\delta(q_3 - q_3')$$

which satisfies eq. 2. Similarly, if the singular point is multiply covered by both $q_1$ and $q_2$ we get

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{\left(\int_{q_{11}}^{q_{12}}\int_{q_{21}}^{q_{22}}h_1\,h_2\,dq_1\,dq_2\right)h_3}\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

## ■ Examples

### ■ Cylindrical coordinates

In cylindrical coordinates at any point $(\rho', \varphi', z')$ except those on the $z$ axis we get

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\,\delta(z-z')$$

and for points on the $z$ axis (which are multiply covered by $\varphi$),

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{2\,\pi\,\rho}\,\delta(\rho)\,\delta(z-z').$$

### ■ Spherical polar coordinates

In spherical polar coordinates at any point $(r', \theta', \varphi')$ except those on the $z$ axis (multiply covered by $\varphi$) and not the origin (multiply covered by $\varphi$ and $\theta$) we have

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi').$$

For points $\vec{r}\,'$ on the positive $z$ axis,

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{2\,\pi\,r^2\sin\theta}\,\delta(r-r')\,\delta(\theta).$$

It is often useful to replace the $\theta$ coordinate with $x=\cos\theta$ in which case we get for a regular point

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{r^2}\,\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')$$

and for a point $\vec{r}\,'$ (except the origin) on the positive $z$ axis

$$\delta^3(\vec{r}-\vec{r}\,')=\frac{1}{2\,\pi\,r^2}\,\delta(r-r')\,\delta(\cos\theta-1).$$

There are similar expressions for points on the negative $z$ axis.

Finally, if the delta function is at the origin $\vec{r}\,'$ you get

$$\delta^3(\vec{r}-\vec{r}\,')=\delta^3(\vec{r})=\frac{1}{4\,\pi\,r^2}\,\delta(r)$$

An important point to notice is that from eq. 2, the dimensions of $\delta^3(\vec{r}-\vec{r}\,')$ is that of the cube of a reciprocal length. In general, a delta function has dimensions which are the reciprocal of its argument. With this, notice that the dimensions are correct in the several representations I have given for $\delta^3(\vec{r}-\vec{r}\,')$. This is a useful little check.

## ■ Poisson's equation for a point charge in curvilinear orthogonal coordinates

It is useful to notice that both the laplacian operator and the delta function have a factor of the reciprocal of the Jacobian in them. Thus in Poisson's equation they can divided out of both sides of the equation.

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \Phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial q_3}\right)\right)$$
$$=-\frac{1}{\varepsilon_0}\frac{1}{h_1 h_2 h_3}\delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

or

$$\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \Phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial q_3}\right)$$
$$=-\frac{1}{\varepsilon_0}\delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

This implies that when you use Poisson's equation to find the Green function in a geometry in orthogonal coordinates, there is never any impediment to integrating over the $\delta$ function and getting a difference of multiples of first derivatives of the function in question. If you have some factor multiplying the derivative that you wish to integrate over an infinitesimal range covering the delta function, you've make a mistake!